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FOR - IAS-Exam.2011

## PHYSICS

### PAPER - I

#### 1. (a) Mechanics of Particles:

Laws of motion; conservation of energy and momentum, applications to rotating frames; centripetal and Coriolis accelerations; Motion under a central force; Conservation of angular momentum, Kepler's laws; Fields and potentials; Gravitational field and potential due to spherical bodies, Gauss and Poisson equations, gravitational self-energy; Two-body problem; Reduced mass; Rutherford scattering; Centre of mass and laboratory reference frames.

#### (b) Mechanics of Rigid Bodies:

System of particles; Centre of mass, angular momentum, equations of motion; Conservation theorems for energy, momentum and angular momentum; Elastic and inelastic collisions; Rigid body; Degrees of freedom, Euler's theorem, angular velocity, angular momentum, moments of inertia, theorems of parallel and perpendicular axes, equation of motion for rotation; Molecular rotations (as rigid bodies); Di-

and tri-atomic molecules; Precessional motion; top, gyroscope.

#### (c) Mechanics of Continuous Media:

Elasticity, Hooke's law and elastic constants of isotropic solids and their interrelation; Streamline (Laminar) flow, viscosity, Poiseuille's equation, Bernoulli's equation, Stokes' law and applications.

#### (d) Special Relativity:

Michelson-Morley experiment and its implications; Lorentz transformations-length contraction, time dilation, addition of relativistic velocities, aberration and Doppler effect, mass-energy relation, simple applications to a decay process; Four dimensional momentum vector; Covariance of equations of physics.

#### 2. Waves and Optics:

##### (a) Waves:

Simple harmonic motion, damped oscillation, forced oscillation and resonance; Beats; Stationary waves in a string; Pulses and wave packets; Phase and group velocities, Reflection and Refraction from Huygens' principle.

##### (b) Geometrical Optics:

Laws of reflection and refraction from Fermat's principle; Matrix method in paraxial optics-thin lens formula, nodal planes, system of two thin lenses, chromatic and spherical aberrations.

##### (c) Interference:

Interference of light-Young's experiment, Newton's rings, interference by thin films, Michelson interferometer; Multiple beam interference and Fabry-Perot interferometer.

#### (d) Diffraction:

Fraunhofer diffraction-single slit, double slit, diffraction grating, resolving power; Diffraction by a circular aperture and the Airy pattern; Fresnel diffraction; half-period zones and zone plates, circular aperture.

#### (e) Polarization and Modern Optics:

Production and detection of linearly and circularly polarized light; Double refraction, quarter wave plate; Optical activity; Principles of fibre optics, attenuation; Pulse dispersion in step index and parabolic index fibres; Material dispersion, single mode fibres; Lasers-Einstein A and B coefficients; Ruby and He-Ne lasers; Characteristics of laser light-spatial and temporal coherence; Focusing of laser beams; Three-level scheme for laser operation; Holography and simple applications.

#### 3. Electricity and Magnetism:

##### (a) Electrostatics and Magnetostatics:

Laplace and Poisson equations in electrostatics and their applications; Energy of a system of charges, multipole expansion of scalar potential; Method of images and its applications; Potential and field due to a dipole, force and torque on a dipole in an external field; Dielectrics, polarization; Solutions to boundary-value problems-conducting and dielectric spheres in a uniform electric field; Magnetic shell, uniformly magnetized sphere; Ferromagnetic materials, hysteresis, energy loss.

##### (b) Current Electricity:

Kirchhoff's laws and their applications; Biot-Savart law, Ampere's law, Faraday's law, Lenz' law; Self and mutual-inductances; Mean and r.m.s values in AC circuits; DC and AC circuits with R, L and C components; Series and parallel resonances; Quality factor; Principle of transformer.

##### (c) Electromagnetic Waves and Black-body Radiation:

Displacement current and Maxwell's equations; Wave equations in vacuum, Poynting theorem; Vector and scalar potentials; Electromagnetic field tensor, covariance of Maxwell's equations; Wave equations in isotropic dielectrics, reflection and refraction at the boundary of two dielectrics; Fresnel's relations; Total internal reflection; Normal and anomalous dispersion; Rayleigh scattering; Blackbody radiation and Planck's radiation law, Stefan-Boltzmann law, Wien's displacement law and Rayleigh-Jeans' law.

#### 4. Thermal and Statistical Physics:

##### (a) Thermodynamics:

Laws of thermodynamics, reversible and irreversible processes, entropy; Isothermal,

adiabatic, isobaric, isochoric processes and entropy changes; Otto and Diesel engines, Gibbs' phase rule and chemical potential; van der Waals equation of state of a real gas, critical constants; Maxwell-Boltzmann distribution of molecular velocities, transport phenomena, equipartition and virial theorems; Dulong-Petit, Einstein, and Debye's theories of specific heat of solids; Maxwell relations and applications; Clausius-Capeyron equation; Adiabatic demagnetisation, Joule-Kelvin effect and liquefaction of gases.

##### (b) Statistical Physics:

Macro and micro states, statistical distributions, Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac distributions, applications to specific heat of gases and blackbody radiation; Concept of negative temperatures.

#### 1. Quantum Mechanics:

Wave-particle duality; Schrodinger's equation and expectation values; Uncertainty principle; Solutions of the one-dimensional Schrodinger equation for a free particle (Gaussian wave-packet), particle in a box, particle in a finite well, linear harmonic oscillator; Reflection and transmission by a step potential and by a rectangular barrier; Particle in a three dimensional box, density of states, free electron theory of metals; Angular momentum; Hydrogen atom; Spin half particles, properties of Pauli spin matrices.

#### 2. Atomic and Molecular Physics:

Stern-Gerlach experiment, electron spin, fine structure of hydrogen atom; L-S coupling, J-J coupling; Spectroscopic notation of atomic states; Zeeman effect; Frank-Condon principle and applications; Elementary theory of rotational, vibrational and electronic spectra of diatomic molecules; Raman effect and molecular structure; Laser Raman spectroscopy; Importance of neutral hydrogen atom, molecular hydrogen and molecular hydrogen ion in astronomy; Fluorescence and Phosphorescence; Elementary theory and applications of NMR and EPR; Elementary ideas about Lamb shift and its significance.

#### 3. Nuclear and Particle Physics:

Basic nuclear properties-size, binding energy, angular momentum, parity, magnetic moment; Semi-empirical mass formula and applications, mass parabolas; Ground state of deuteron, magnetic moment and non-central forces; Meson theory of nuclear forces; Salient features of nuclear forces; Shell model of the nucleus - successes and limitations; Violation of parity in beta decay; Gamma decay and internal conversion; Elementary ideas about Mossbauer spectroscopy; Q-value of nuclear reactions; Nuclear fission and fusion, energy production in stars; Nuclear reactors.

Classification of elementary particles and their interactions, Conservation laws; Quark structure of hadrons, Field quanta of electroweak and strong interactions; Elementary ideas about unification of forces; Physics of neutrinos.

#### 4. Solid State Physics, Devices and Electronics:

Crystalline and amorphous structure of matter; Different crystal systems, space groups; Methods of determination of crystal structure; X-ray diffraction, scanning and transmission electron microscopies; Band theory of solids - conductors, insulators and semiconductors; Thermal properties of solids, specific heat, Debye theory; Magnetism: dia, para and ferromagnetism; Elements of superconductivity, Meissner effect, Josephson junctions and applications; Elementary ideas about high temperature superconductivity.

Intrinsic and extrinsic semiconductors; p-n-p and n-p-n transistors; Amplifiers and oscillators, Op-amps, FET, JFET and MOSFET; Digital electronics-Boolean identities, De Morgan's laws, logic gates and truth tables; Simple logic circuits; Thermistors, solar cells; Fundamentals of microprocessors and digital computers.

# MECHANICS

# Physics 1

## Basics

Lectures -

Chapters 1, 2 of D.S. Mathur

## Mechanics of Particles

Lectures 1, 2, 3, 4, 5, 6, 7, 8

Tut 1, 2, 3

Chapters 5, 6, 11 of D.S. Mathur

## Mechanics of Rigid Bodies

Lectures 9, 10

Tut 4

Chapters 10 of D.S. Mathur

## Mechanics of Continuous Media

Lectures 11, 12

Tut 5

Chapters 12, 14 of D.S. Mathur

## Relativity

Lectures 13, 14, 15, 16, 17

Tut 6

Chapters 3 of D.S. Mathur

14/11/11

# Paper 1

300 Marks : 180 minutes

10 Marks : 6 minutes

1 minute for thinking

5 minutes  $\Rightarrow$  1 Page

$$\begin{aligned} \text{⊛ kinetic Energy for a rotating system} \\ &= \frac{1}{2} \vec{\omega} \cdot \vec{J} \\ &= \frac{1}{2} \vec{\omega} \cdot [I \vec{\omega}] \end{aligned}$$

$$\text{⊛ } G = 6.67 \times 10^{-11} \text{ SI.}$$

**DO NOT WRITE MORE THAN ASKED**

$\Rightarrow$  **10 Marks : 1 Page in 5 minutes**

## Section A

① Mechanics

② Optics

Q1: half from ①  
half from ②

## Section B

③ Electricity & Magnetism

④ Heat & Thermodynamics

Q5: half from ③  
half from ④

$\rightarrow$  Hence whole course needs to be done for full attempt

But Master 2 courses out of 4.

[to attempt any question on that topic]

○ Never interact with unsuccessful candidates

○ 2 common mistakes

(i) Lack of proper strategy → lack of proper study

(ii) Lack of proper practice → not to-the-point answers

Correct answer can fetch ~~20~~ 50% to 75% depending upon 'Quality of correct answer'.

Section A : Mechanics

Optics

Q1 6x 10 (3, 3)

Q2 Mechanics

Q3 Mechanics & Optics

Q4 Optics

① Physics लिखनी होती है !!

② जो पूछा है, उसे define करो !!

○ Prepare 3 books thoroughly for each Paper to attempt minimum 270 marks.



Mechanics : 120 Marks in Paper  $\equiv$  2 Questions

4 units in syllabus :

- ① Particle Dynamics / System of Particles
- ② Rigid Body Dynamics / System of Particles
- ③ Mechanics of Continuous Media
- ④ Special theory of Relativity

every unit is important....

WE CANNOT SKIP ANY SUBTOPIC

MECHANICS

★ For integration by parts, 2<sup>nd</sup> function should follow the order I L A T E

with exponential being the 1<sup>st</sup> preference for 2<sup>nd</sup> function.

Inverse log Arith trig exp

(Write the two functions in this order... automatically the function of right becomes the 2<sup>nd</sup> function)

# Particle Dynamics

- ① Conservation laws, Elastic & Inelastic Collisions, Rocket Motion
- ② System of Particles  
Centre of Mass  
Transformation of physical quantities from lab frame to Centre of Mass frame.
- ③ Rutherford scattering, Differential Scattering Cross-section
- ④ Rotating Frame of reference: Coriolis & Centrifugal terms
- ⑤ Gravitation
- ⑥ Central Force Problems

6 chapters in 'Particle dynamics' unit.

⊙ Classical Mechanics ... J.C. Upadhyay

⊙ Theoretical Mechanics ... M.R. Spigal

X D.S. Mathur

⇒ General Mechanics + Classical Mechanics : hence not everything in 1 book

# Event

# MECHANICS (1)

15/11/2011

- $\vec{v}$  &  $\vec{a}$  in polar form
- Conservation of energy
- Centre of Mass Project
- 1-d & 2-d collision
- Scattering Cross section

✓ Specifying space and time determines event.

✓ These are specified wst. a frame of reference.

✓ There are 2 types of frames of reference:

- 1) Inertial Frame : state of observer does not change (non accelerated)
- 2) Non Inertial Frame : if state of observer change (accelerated frame)

→ Rocket Motion

## Frame of Reference

### Inertial Frame

- $\vec{v} = \text{constant}$
- $\frac{d\vec{v}}{dt} = 0$
- state of observer remains constant

### Non Inertial Frame

- $\vec{v} \neq \text{constant}$
- accelerated motion
- $\frac{d\vec{v}}{dt} \neq 0$

★ All Basic physical laws hold good in inertial frame of reference

## Laws of Motion

→ Single Particle : dimensions of the particle are insignificant to the distances being talked about

✓ Point particle can have mass as well as charge.

✓ Interaction of 2 particles in 4 ways only:

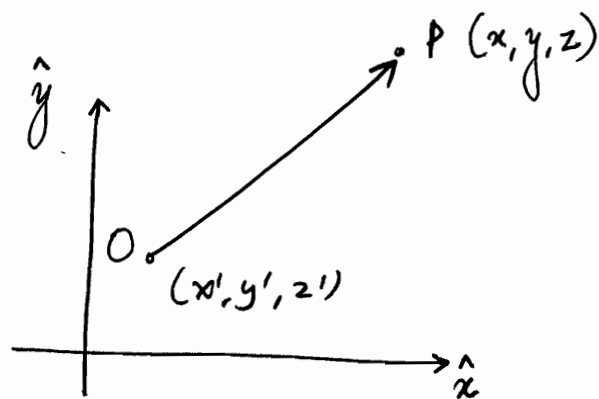
(1) By Virtue of mass : Gravitational interaction  
(Mechanics)

(2) By virtue of charge : Electro magnetic interaction  
(Electricity & Magnetism)

(3) Strong Interaction : Nuclear interaction  
(short time required for huge force)

(4) Weak Interaction : long time interaction

✓ When we study collision, the interaction force for the 2 particles will be of same kind. Not that one interacting due to mass, other interacting due to charge.



$\vec{OP}$  = position vector of Particle P w.r.t. Observer O

$$= \vec{r}$$

$$= (x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k}$$

○ We have

- Addition
- Subtraction
- Dot Product
- Cross Product
- Multiplication with scalar

of vectors.

No multiplication or division.

$$\vec{r} = r \hat{r}$$

↑  
unit vector along  $\vec{r}$



$$\hat{r} = \left( \frac{\vec{r}}{r} \right)$$

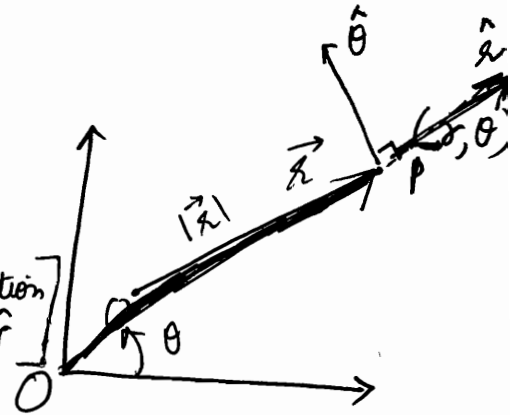
$$r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

usually 0 is assumed as (0,0,0)

Note that  $\theta$  of 2-d Polar coordinate is completely different from  $\theta$  of spherical coordinates.

$$\Rightarrow r = \sqrt{x^2 + y^2 + z^2}$$

Perpendicular to  $\hat{r}$  is  $\hat{\theta}$  [ $\hat{\theta}$  is addition of  $90^\circ$  to  $\hat{r}$ ]



$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

⊛ Unlike cartesian directions ( $\hat{x}, \hat{y}, \hat{z}$ ), polar directions ( $\hat{r}, \hat{\theta}$ ) are not fixed w.r.t. time, hence they also vary with time thereby having their change's contribution too in derivative of a vector quantity....

### Law 1 Law of Inertia

w/o use of external force, state of Particle P does not change.

### Law 2

$$\begin{aligned} \vec{F}_{\text{ext}} &= \frac{d\vec{p}}{dt} \\ \text{definition of Force} &= \frac{d(m\vec{v})}{dt} = m \left( \frac{d\vec{v}}{dt} \right) = m\vec{a} \end{aligned}$$

(assuming  $\frac{dm}{dt} = 0$ )

We know,

$$\vec{v} = \left( \frac{d\vec{r}}{dt} \right)$$

$$\vec{a} = \left( \frac{d\vec{v}}{dt} \right) = \left( \frac{d^2\vec{r}}{dt^2} \right)$$

← Perfectly alright. We have not yet used any coordinates. Its written in vector form.

1<sup>st</sup> law is derived from 2<sup>nd</sup> law.

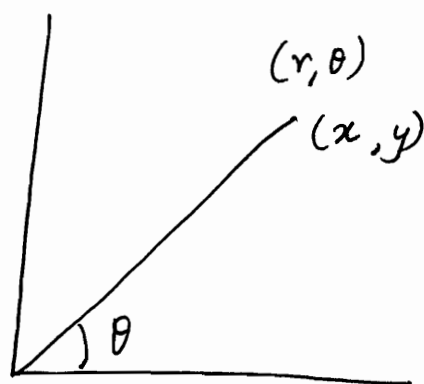
It is preferable to write in vector as it is independent of system of coordinates

$$\vec{F}_{\text{ext}} = m \vec{a}$$

It is valid in all systems of coordinates.

$$\begin{aligned} \vec{r} &= x \hat{i} + y \hat{j} \\ &= r \cos \theta \hat{i} + r \sin \theta \hat{j} \\ &= r (\cos \theta \hat{i} + \sin \theta \hat{j}) \end{aligned}$$

$$\hat{r} = (\cos \theta \hat{i} + \sin \theta \hat{j})$$



$$\vec{a} = a_r \hat{r} + a_\theta \hat{\theta}$$

$$F_{\text{ext}}(r) = m a_r$$

$$F_{\text{ext}}(\theta) = m a_\theta$$

$$F_{\text{ext}}(\phi) = m a_\phi$$

$$\begin{aligned} F_{\text{ext}} &= F_{\text{ext}}(r) \hat{r} \\ &+ F_{\text{ext}}(\theta) \hat{\theta} \\ &+ F_{\text{ext}}(\phi) \hat{\phi} \end{aligned}$$

### Law 3

$$F_{ij} + F_{ji} = 0$$

Every action has equal and opposite reaction.

$$\vec{F}_{\text{ext}} = F_{ij} + F_{ji}$$

$$\text{If } \vec{F}_{\text{ext}} = 0$$

$$\Rightarrow F_{ij} + F_{ji} = 0$$

$$\vec{F}_{\text{ext}} = \left( \frac{d\vec{p}}{dt} \right)$$

$$\vec{F}_{\text{ext}} = m\vec{a} \quad (\text{if mass is constant})$$

In  $(r, \theta)$  coordinates,

$$\vec{v} = v_r \hat{r} + v_\theta \hat{\theta}$$

$$\vec{v} = \dot{r} \hat{r} + r\dot{\theta} \hat{\theta}$$

$$\vec{a} = a_r \hat{r} + a_\theta \hat{\theta}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2) \hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta}) \hat{\theta}$$

$$\vec{r} = x\hat{i} + y\hat{j} = r(\cos\theta \hat{i} + \sin\theta \hat{j}) = r\hat{r}$$

$$\hat{r} = \cos\theta \hat{i} + \sin\theta \hat{j}$$

$$\hat{\theta} = -\sin\theta \hat{i} + \cos\theta \hat{j}$$

$$\hat{\theta} = \cos(90+\theta) \hat{i} + \sin(90+\theta) = -\sin\theta \hat{i} + \cos\theta \hat{j}$$



$$\vec{v} = \frac{d\vec{r}}{dt}$$

All the dots ( $\dot{r}$  etc.) are time derivatives

$$= \frac{d}{dt} (r \hat{e})$$

$$= \left(\frac{dr}{dt}\right) \hat{e} + r \left(\frac{d\hat{e}}{dt}\right)$$

$$\underline{\underline{\vec{v} = \dot{r} \hat{e} + r \dot{\theta} \hat{\theta}}} \quad \text{--- (1)}$$

In a special case of circular motion, no radial velocity, only tangential velocity because  $\dot{r} = 0$

$$\frac{d\hat{e}}{dt} = \frac{d}{dt} (\cos\theta \hat{i} + \sin\theta \hat{j})$$

$$= -\sin\theta \dot{\theta} \hat{i} + \cos\theta \dot{\theta} \hat{j}$$

$$= (-\sin\theta \hat{i} + \cos\theta \hat{j}) \dot{\theta}$$

$$\boxed{\left(\frac{d\hat{e}}{dt}\right) = \dot{\theta} \hat{\theta}}$$

$$\frac{d\hat{\theta}}{dt} = (-\cos\theta \hat{i} - \sin\theta \hat{j}) \dot{\theta}$$

$$\boxed{\left(\frac{d\hat{\theta}}{dt}\right) = -\dot{\theta} \hat{e}}$$

Using (1)

$$\vec{a} = \left(\frac{d\vec{v}}{dt}\right) = \left(\frac{d\dot{r}}{dt}\right) \hat{e} + \dot{r} \left(\frac{d\hat{e}}{dt}\right)$$

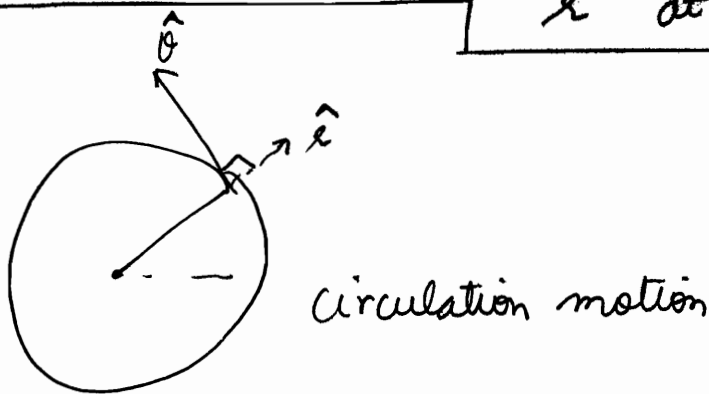
$$+ \frac{d}{dt} (r \dot{\theta}) \hat{\theta} + r \dot{\theta} \left(\frac{d\hat{\theta}}{dt}\right)$$

$$= \underbrace{\ddot{r} \hat{e} + \dot{r} \dot{\theta} \hat{\theta}} + \underbrace{r \dot{\theta}^2 \hat{e} + \frac{d}{dt} (r \dot{\theta}) \hat{\theta}}$$

$$= \underbrace{\dot{r} \dot{\theta} \hat{\theta} + \ddot{r} \hat{e}} + \underbrace{r \dot{\theta}^2 \hat{e} + (\dot{r} \dot{\theta} + r \ddot{\theta}) \hat{\theta}}$$

$$= \underline{\underline{(\ddot{r} - r \dot{\theta}^2) \hat{e} + (2\dot{r} \dot{\theta} + r \ddot{\theta}) \hat{\theta}}}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2) \hat{r} + \frac{1}{r} \frac{d}{dt} (r^2\dot{\theta}) \hat{\theta}$$



⊙ All the results derived till now are for a general motion in a 2-d plane for Polar ~~the~~ coordinates.

$r = \text{constant}$

$\dot{r} = 0$

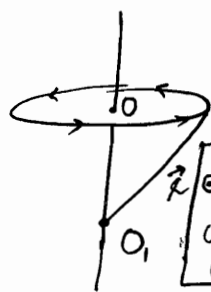
$$\Rightarrow \vec{v} = r\dot{\theta} \hat{\theta}$$

$$= \omega r \hat{\theta}$$

$$= [\vec{\omega} \times \vec{r}]$$

⊙ Polar Coordinates is a 2-d  $(r, \theta)$  coordinate system

$$(\dot{\theta} = \frac{d\theta}{dt} = \omega)$$



⊙  $[\vec{\omega} \times \vec{r}]$  is a general result in which  $\vec{r}$  can be measured from any origin  $O_1$  lying on axis of rotation!!

⊙ Cartesian Coordinate system  $(x, y, z)$ .

In the simplest case  $O_1$  is taken at  $O$  i.e. along axis in the plane of rotation motion.

$$\vec{a} = [0 - \omega^2 r] \hat{r} + [0 + r\ddot{\theta}] \hat{\theta}$$

$$(\ddot{\theta} = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \alpha)$$

$$= -\omega^2 r \hat{r} + \alpha r \hat{\theta}$$

In UNIFORM circulation motion i.e.  $\alpha = 0$

$$\vec{a} = -\omega^2 r \hat{r}$$

Since direction is towards  $(-\hat{r})$ , its called Centripetal acceleration, i.e. directed towards centre.

⊙ Note that  $v$  ~~is~~ but  $|\vec{a}| \neq 0$ ; due to  $|\vec{v}| = \text{const}$

constantly changing direction in uniform circular motion.

### Conservation Laws

+5: Charge,  $L_x, L_y, L_z, B$

Total: 8

### 3 conservation laws:

(1) Conservation of linear momentum

if  $\vec{F}_{\text{ext}} = 0 \Rightarrow \frac{d\vec{P}}{dt} = 0 \Rightarrow \vec{P}_{\text{system}}$  is constant

We know, Torque

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Equivalent of Force in rotational motion is Torque

$$\vec{\tau} = \vec{r} \times \frac{d\vec{P}}{dt} \quad \xrightarrow{\text{def'n of } \tau} \quad \boxed{\vec{\tau} = \frac{d\vec{J}}{dt}}$$

$$= \frac{d}{dt} (\vec{r} \times \vec{P}) = \frac{d}{dt} \vec{J}$$

$$\left[ = \frac{d\vec{r}}{dt} \times \vec{P} + \vec{r} \times \frac{d\vec{P}}{dt} \right]$$

$\frac{d\vec{r}}{dt} \times \vec{P} = 0$  (They both are parallel)

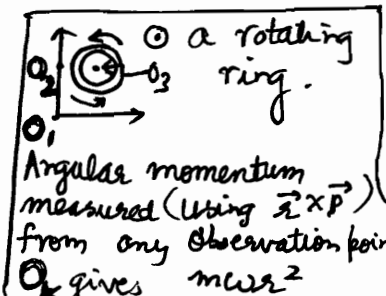
~~or for a body rotational system,  $\vec{r}$  is a point.~~

$$\vec{\tau} = \vec{r} \times \vec{F} = \frac{d\vec{J}}{dt}$$

$$\boxed{\vec{J} = \vec{r} \times \vec{P}}$$

(2) Conservation of angular momentum

if  $\tau_{\text{ext}} = 0 \Rightarrow \vec{J} = \text{const.}$



### (3) Conservation of Mechanical Energy

Conservative system means mechanical energy = const.

$$\text{or } K.E + P.E = \text{Const}$$

$$\text{or } \boxed{T + U = \text{Const.}}$$

- ⊙  $U + T = \text{Const}$
- ⊙  $\nabla \times \vec{F} = 0$
- ⊙  $\vec{F} = -\nabla U$
- ⊙  $\int_a^b W$  : Path independent

For non conservative or dissipative systems

$$T + U \neq \text{Const.}$$

Forces working in Conservative systems are Conservative forces. 2 basic Properties of a conservative force.

⊙ Work done by conservative forces is Path independent.

⊙ Mathematically, <sup>Conservative</sup> Force should be able to be represented as gradient of some scalar function.

$$\boxed{\vec{F}_{\text{conservative}} = -\nabla \phi}$$

$\phi$  will be called Potential

minus is included, so that,  $T + U = \text{Const}$  can be achieved :)  
⊙ signifies that if Force does a work, P.E. reduces

⊙  $\nabla$  is pronounced as 'DEL'

$$dW = \vec{F} \cdot d\vec{r} = -\nabla \phi \cdot d\vec{r}$$

$$= -\left( \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right) \cdot (dx \hat{i} + dy \hat{j} + dz \hat{k})$$

$$= -\left( \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right) = -(d\phi)$$

$$\int_1^2 \vec{F} \cdot d\vec{x} = - \int_1^2 d\phi$$

$$= \phi(1) - \phi(2)$$

$$\Rightarrow \int_1^2 m \frac{d\vec{v}}{dt} \cdot d\vec{x} = U(1) - U(2) \quad (\text{Change of notation})$$

$$\star \left( \frac{dv_x}{dt} \hat{i} + \frac{dv_y}{dt} \hat{j} + \frac{dv_z}{dt} \hat{k} \right) \cdot (dx \hat{i} + dy \hat{j} + dz \hat{k})$$

$$\Rightarrow \int_1^2 m d\vec{v} \cdot \vec{v} = U(1) - U(2) \quad \left[ \begin{array}{l} \frac{d\vec{v}}{dt} = \frac{d\vec{v}}{d\vec{x}} \frac{d\vec{x}}{dt} \\ \text{chain rule} \end{array} \right]$$

$$= \sum \left( \frac{dv_x}{dx} * \frac{dx}{dt} \hat{i} \right) \cdot (dx \hat{i})$$

$$= \sum \frac{dv_x}{dx} * \frac{dx}{dt} * dx = \sum v_x dv_x = \left( v_x \hat{i} + v_y \hat{j} + v_z \hat{k} \right) \cdot \left( dv_x \hat{i} + dv_y \hat{j} + dv_z \hat{k} \right)$$

$$\Rightarrow \int_1^2 m \frac{1}{2} (d(\vec{v} \cdot \vec{v})) = U(1) - U(2) = \underline{\underline{\vec{v} \cdot d\vec{v}}}$$

$$\Rightarrow \int_1^2 d \left( \frac{1}{2} m v^2 \right) = U(1) - U(2)$$

$$\Rightarrow \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 = U(1) - U(2)$$

$$\Rightarrow T_2 - T_1 = U(1) - U(2)$$

$$\Rightarrow U(1) + T(1) = U(2) + T(2)$$

$$\begin{array}{l} d(\vec{v} \cdot \vec{v}) \\ d\vec{v} \cdot \vec{v} + \vec{v} \cdot d\vec{v} \\ = 2(\vec{v} \cdot d\vec{v}) \end{array}$$

Hence, if  $\vec{F} = -\vec{\nabla} U$

$$\Rightarrow T + U = \text{const.}$$

• curl of gradient = 0

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{F}_{\text{conservative}} = 0}$$

Note that curl of  $\vec{F}$  is 0  
This is necessary and sufficient condition for a conservative field of force  $F$ .

•  $\oint \vec{F} \cdot d\vec{r} = 0 = \oint dU$

$$\left. \begin{array}{l} \vec{F} = -\vec{\nabla} U \\ \vec{\nabla} \times \vec{F} = 0 \end{array} \right\} \Rightarrow \vec{F} \cdot d\vec{r} = -dU$$

$$\int_1^2 \vec{F} \cdot d\vec{r} = U(1) - U(2)$$

$$U(r) - U(\text{ref}) = - \int_{\text{ref}}^r \vec{F} \cdot d\vec{r}$$

if ref at  $\infty$  and  $U(\text{ref}) = 0$

$$\Rightarrow \boxed{U(r) = - \int_{\infty}^r \vec{F} \cdot d\vec{r}}$$

★ Not that ref can be at 0. Then also it's perfectly alright. All the Potentials will then be readjusted....

$[-F]$  represents the external force

$\Rightarrow U(r)$  is the work done by an external force in bringing a particle from  $\infty$  to  $r$ .

Q 29 / 2008 / 10 marks

$$\vec{F} = (2xy + z^3) \hat{i} + x^2 \hat{j} + 3xz^2 \hat{k}$$

Is it a conservative field?

If yes, what is scalar potential?

$$\underline{\text{A29}}) \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= (2x - 2x) \hat{k} + (3z^2 - 3z^2) \hat{j} + (0) \hat{i}$$

$$= 0$$

Hence conservative force.

$$\vec{F} = -\nabla U \quad \text{or} \quad dU = -\vec{F} \cdot d\vec{r}$$

$$\Rightarrow \underline{F_x = -\left(\frac{\partial U}{\partial x}\right)}$$

$$\Rightarrow U = -\int F_x dx + C_1(y, z)$$

$$= -(x^2y + xz^3) + C_1(y, z)$$



$$U = - \int F_3 dy + C_2(x, z)$$
$$= -x^2y + C_2(x, z)$$

$$U = - \int F_2 dz + C_3(x, y)$$
$$= -xz^3 + C_3(x, y)$$

$$U = - (x^2y + xz^3)$$

$$C_1(y, z) = 0$$

$$C_2(x, z) = -xz^3$$

$$C_3(x, y) = -x^2y$$

# Gradient

HW

Suppose we have a scalar function (eg. Temperature) that depends upon 3 space coordinates.

How does  $T$  change when we change one or more of those variables?

The change of  $T$ , <sup>per unit change in variables</sup> is called gradient at position  $(x, y, z)$ . This change will of course depend upon the direction in which we move. Hence gradient of  $T$  is a vector function

Gradient of  $T$  or  $\vec{\nabla}T$  is thus a vector.

Now if we differentiate  $T$  w.r.t  $x$ , that tells us the "change of  $T$ " in the  $x$  direction. This is, hence, the  $i^{\text{th}}$  component of gradient of  $T$ .

Hence, the definition of gradient of  $T$  or  $\nabla T$  is

$$\vec{\nabla}T = \left(\frac{\partial T}{\partial x}\right) \hat{i} + \left(\frac{\partial T}{\partial y}\right) \hat{j} + \left(\frac{\partial T}{\partial z}\right) \hat{k}$$

Change in Temperature  $dT = (\vec{\nabla}T) \cdot d\vec{l}$

There are 2 important results:

- The gradient  $\vec{\nabla}T$  points in the direction of maximum increase of the function  $T$ .
- The magnitude  $|\vec{\nabla}T|$  gives the slope / rate of increase along this maximal direction

HW

$$\odot \frac{d}{dt} (\vec{a} \cdot \vec{b}) = \frac{d}{dt}(\vec{a}) \cdot \vec{b} + \vec{a} \cdot \frac{d}{dt}(\vec{b})$$

MATHS

$$\odot \frac{d}{dt} (\vec{a} \times \vec{b}) = \frac{d}{dt}(\vec{a}) \times \vec{b} + \vec{a} \times \frac{d}{dt}(\vec{b})$$

$$\odot \vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$$

$$\odot \vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} (\vec{a} \cdot \vec{c}) - \vec{c} (\vec{a} \cdot \vec{b})$$

↑  
(BC... सत्य में है)

(BAC minus CAB)

$$\odot A \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

A theorem on Partial derivatives states :-

$$\odot dT = \left( \frac{\partial T}{\partial x} \right) dx + \left( \frac{\partial T}{\partial y} \right) dy + \left( \frac{\partial T}{\partial z} \right) dz$$

where  $T = f(x, y, z)$

$$\odot \vec{\nabla} \cdot (\vec{A} + \vec{B}) = \vec{\nabla} \cdot \vec{A} + \vec{\nabla} \cdot \vec{B}$$

$$\vec{\nabla} \times (\vec{A} + \vec{B}) = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{B}$$

Vector algebra is distributive but non associative.

Curvilinear Coordinates (Polar, Spherical, Cylindrical) are a little tricky to handle mathematically as their unit vectors are not fixed. Unit vectors depend upon the position and hence they too need to be differentiated.

# Angular Momentum

HW

⊙ Angular Momentum or Rotational Momentum is a "conserved vector" quantity that can be used to describe the overall state of a physical system.

Angular Momentum ' $L$ ' of a particle wrt. some point of observation is

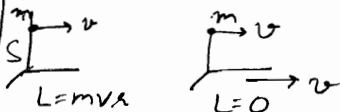
$$L = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

where,

$\vec{p}$ : linear momentum ( $m\vec{v}$ )

$\vec{r}$ : particle's position from observation point

① Every thing depends on frame of reference



② अगर  $\vec{r}$  frame  $\vec{r}$   
 $\tau = 0 \Rightarrow L$  is conserved

अगर  $\vec{r}$  frame of reference  $\vec{r}$   $\tau \neq 0 \Rightarrow$   
 $L$  of same particle is not conserved & changes

Angular Momentum of a system of particles (eg. a rigid body) is the sum of angular momenta of the individual particles.

③ If observation is different in different frame, then which frame is correct?? No frame is right or wrong. We choose that frame from where observation is simplest.....

⊙ A special case is a body rotating around an axis of symmetry; angular momentum can be expressed, (we can derive it from  $\int \vec{r} \times d\vec{p}$ ) as product of body's moment of inertia  $I$  (a measure of object's resistance to change in its rotation rate) and its angular velocity  $\vec{\omega}$ .

$$\vec{L} = I\vec{\omega}$$

HW  $\odot$  Angular momentum is conserved in a system where there is no net external torque.

$\odot$  Although we can choose any point of reference, its mathematically simple to choose Centre of Mass as the observation point to consider the angular momentum of a system of particles.

## Torque

As we defined Force after defining momentum as

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Similarly we define Torque as

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt} (\vec{r} \times \vec{p})$$

$$= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

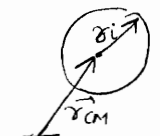
$$= \vec{v} \times m\vec{v} + \vec{r} \times \frac{d\vec{p}}{dt}$$

= 0

$$= \vec{r} \times \vec{F}$$

Hence

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$



$\vec{r}_{CM}$

$$\begin{aligned} \vec{L} &= \sum m_i \vec{r}_i \times \vec{v}_i \\ &= \sum m_i (\vec{r}_{CM} + \vec{r}_i) \times (\vec{v}_{CM} + \vec{v}_i) \\ &= \sum m_i (\vec{r}_{CM} \times \vec{v}_{CM}) \\ &\quad + \sum m_i \vec{r}_{CM} \times \vec{v}_i \\ &\quad + \sum m_i \vec{r}_i \times \vec{v}_{CM} \\ &\quad + \sum m_i \vec{r}_i \times \vec{v}_i \\ &= L_{CM, g} + L_{b, CM} \end{aligned}$$

$\odot$  Note that  $\vec{L}_{1,2} = \vec{L}_{1,3} + \vec{L}_{3,2}$  is not valid in general. Rather its not valid in any vector unless its of the form  $\frac{d^n \vec{r}}{dt^n}$  ( $n=0,1,2$ )

But as a special case for a rigid body:  $\vec{L}_{b,g} = \vec{L}_{b,cm} + \vec{L}_{cm,g}$

# More Maths

HW

⊙ Equality of cross derivatives for Path Independent variable  $T$

$$\frac{\partial}{\partial x} \left( \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial T}{\partial x} \right) \quad \left[ \text{Complete differential} \right]$$

$$\star \int_a^b T \, dx = T(b) - T(a)$$

⊙ **Laplacian** or  $\nabla^2 T$  is defined as 2<sup>nd</sup> order derivative  
divergence of a gradient:  $\vec{\nabla} \cdot (\vec{\nabla} T)$   
It is a scalar quantity scalar  $\nabla^2$  scalar

$$\begin{aligned} \nabla^2 T &= \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot \left( \hat{x} \frac{\partial T}{\partial x} + \hat{y} \frac{\partial T}{\partial y} + \hat{z} \frac{\partial T}{\partial z} \right) \\ (\text{scalar}) &= \left[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] \end{aligned}$$

⊙ Curl of a gradient  $\vec{\nabla} \times (\vec{\nabla} T) = 0$

⊙ Divergence of a curl  $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

## Fundamental Theorems of Integrals

1)  $\int_a^b \left( \frac{df}{dx} \right) dx = f(b) - f(a)$  Fundamental theorem of calculus

2)  $\int_a^b (\vec{\nabla} T) \cdot d\vec{l} = T(b) - T(a)$  Fundamental theorem of gradients

3)  $\int_{\text{volume}} (\nabla \cdot \vec{A}) dV = \oint_{\text{surface}} \vec{A} \cdot d\vec{a}$  Fundamental theorem of divergence or GAUSS THEOREM

4)  $\int_{\text{surface}} (\nabla \times \vec{A}) \cdot d\vec{a} = \oint_{\text{Perimeter}} \vec{A} \cdot d\vec{l}$  Fundamental theorem of curls or STOKES THEOREM

State function : difference b/w 2 states is important  
 Absolute value at a given state can be arbitrary, depending on point of observation.

$$Q/ \quad \vec{F} = (y^2 - 2xy z^3) \hat{i} + (3 + 2xy - x^2 z^3) \hat{j} \\ + (6z^3 - 3x^2 y z^2) \hat{k}$$

Is it Conservative??

Find Potential . Find out work done

$$A/ \quad F = -\vec{\nabla} U$$

$$F_x = -\frac{\partial U}{\partial x} \Rightarrow U = -\int F_x dx + C(y, z)$$

$$-U = y^2 x - y z^3 x^2 + C_1(y, z)$$

$$-U = 3y + x y^2 - x^2 z^3 y + C_2(y, z)$$

$$-U = \frac{3z^4}{2} + x^2 y z^3 + C_3(x, y)$$

$$U = -x y^2 + x^2 z^3 y + 3y + \frac{3}{2} z^4 + C$$

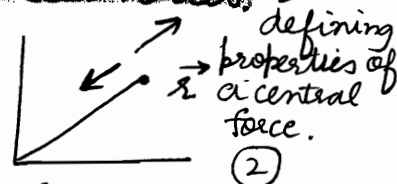
$$\text{Work done} = \int \vec{F} \cdot d\vec{e} = \int -\vec{\nabla} U \cdot d\vec{e} = -\int dU \\ = U_1 - U_2$$



whose magnitude is  $\frac{1}{\text{distance}}$  as measured from a fixed point, those forces are called Central Forces. ①

$$\vec{F} = F(r) \hat{r} \quad \left[ \begin{array}{l} \text{no dependence} \\ \text{on } \theta \text{ or } \dot{r} \end{array} \right]$$

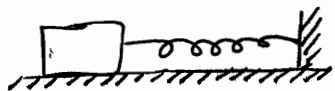
Such forces can be written as  $-\vec{\nabla} U$   
i.e. they are conservative forces.



② Central Force is directed along the line joining the fixed point and the object.

③ Non-Conservative Force: Mechanical Energy is dissipated

### Example of Conservative System of a Central Force.



Conservative force

$$\vec{F} = -k\vec{x}$$

$$\ddot{x} + \left(\frac{k}{m}\right)x = 0$$

let  $\sqrt{\frac{k}{m}} = \omega$

$$\Rightarrow \ddot{x} + \omega^2 x = 0$$

$$\Rightarrow x = A \sin(\omega t + \phi) \quad \text{or } x = x_{\max} \sin(\omega t + \phi)$$

$$\Rightarrow \dot{x} = \omega A \cos(\omega t + \phi)$$

$$E = T + U = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$

$$= \frac{1}{2} m \omega^2 A^2 \cos^2(\omega t + \phi) + \frac{1}{2} k A^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2} m \omega^2 A^2$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$= \frac{1}{2} k A^2 \quad \left( \text{Total energy is independent of mass} \right)$$

✓ All resistive forces that resist free motion, like friction, are non conservative forces.

✓ If Force is dependent on velocity (eg. when friction restricts the motion)

i.e.  $\vec{F} = f(r, \dot{r})$

Now, the  $\vec{F}$  is no longer conservative. We cannot write it as gradient of a scalar potential.

Medium resistances are functions of velocity typically.

$$\vec{F}_{\text{resistand}} \propto -\vec{v}$$

$$\vec{F}_{\text{resistive}} = -b\vec{v}$$

$b$ : damping const.

$$b = \left( \frac{F}{v} \right)$$

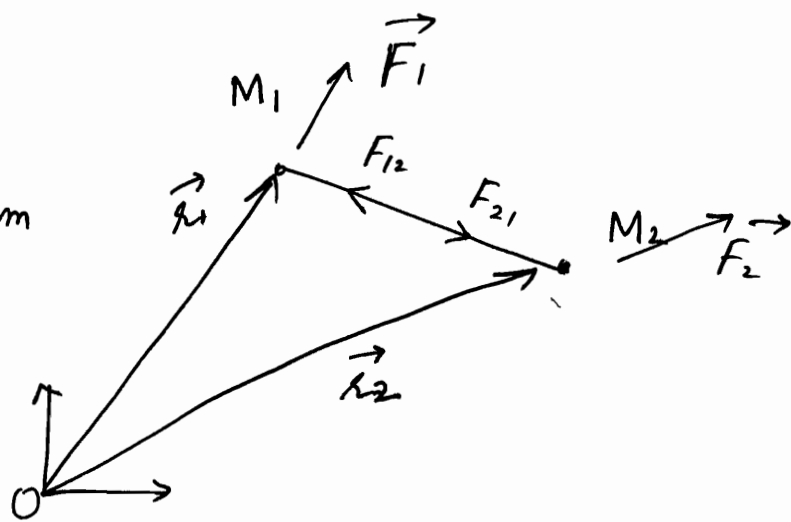
eg. Stokes law

### Centre of Mass

Let us consider a simple system of 2 particles or point masses.

$$m_1 \frac{d^2 \vec{r}_1}{dt^2} = \vec{F}_1 + \vec{F}_{12}$$

[ Force on 1 due to 2 ]



$$m_2 \frac{d^2 \vec{r}_2}{dt^2} = \vec{F}_2 + \vec{F}_{21}$$

$$\frac{d^2}{dt^2} (m_1 \vec{r}_1) = \vec{F}_1 + \vec{F}_{12}$$

$$\frac{d^2}{dt^2} (m_2 \vec{r}_2) = \vec{F}_2 + \vec{F}_{21}$$

Adding ① and ②

$$\frac{d^2}{dt^2} (m_1 \vec{r}_1 + m_2 \vec{r}_2) = \vec{F}_1 + \vec{F}_{12} + \vec{F}_2 + \vec{F}_{21}$$

We know,  $\vec{F}_{12} + \vec{F}_{21} = 0$ , according to Newton's 3<sup>rd</sup> law

$$\Rightarrow \frac{d^2}{dt^2} (m_1 \vec{r}_1 + m_2 \vec{r}_2) = \vec{F}_1 + \vec{F}_2 = \vec{F}_{\text{ext}}$$

$$\vec{F}_{\text{ext}} = M \frac{d^2 \vec{R}}{dt^2}$$

$\vec{R}$  is that point where whole  $\vec{F}_{\text{ext}}$  can be assumed to act for the whole system

$$\Rightarrow \frac{d^2}{dt^2} [m_1 \vec{r}_1 + m_2 \vec{r}_2] = \frac{d^2}{dt^2} [(m_1 + m_2) \vec{R}]$$

$$\Rightarrow \vec{R} = \left( \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \right)$$

If there are  $k$  such particles (discrete point masses)

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$$

← Always write the results in vector form

What if uniform distribution of mass?

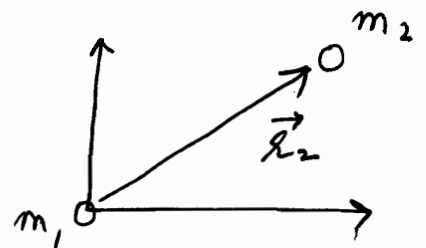
$$\vec{R} = \frac{\int \vec{r} \, dm}{\int dm}$$

Single integral is just notational, it can be single, double or triple integral depending upon distribution of mass.

- Centre of Mass of a system of Particles is the point
- where the net external force acts and
  - where the total mass of the system is concentrated
- (Note that both the points are essential for complete def<sup>n</sup>)

### Properties of Centre of Mass

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{(m_1 + m_2)}$$



If I keep observation point at  $m_1$ , i.e.  $r_1 = 0$

$$\vec{R} = \frac{m_2 \vec{r}_2}{(m_1 + m_2)} = \frac{\vec{r}_2}{1 + \left(\frac{m_1}{m_2}\right)}$$

If  $m_1 \neq m_2$ ,  $\vec{R} = \frac{\vec{r}_2}{2}$  (mid point)

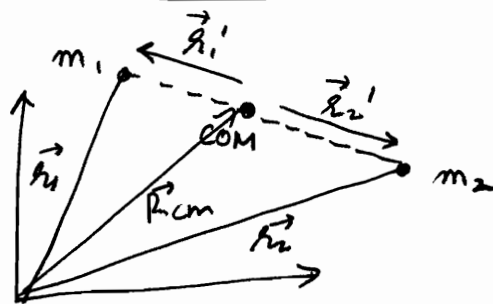
✓ If  $m_1 \gg m_2$ ,  $\vec{R} \approx 0$  (Centre of Mass shifts towards heavier mass)

Hence, we try to choose the centre of coordinates near the heavy bodies to calculate Centre of Mass

✓ Centre of Mass of 2 particle system lies on line joining the 2 particles i.e. collinear property.

We need to prove:  $\vec{r}_1' + \lambda \vec{r}_2' = 0$

$\vec{r}_1'$  : position of 1st particle wrt. Centre of mass.



$$\vec{R}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\vec{r}_1' = \vec{r}_1 - \vec{R}_{cm}$$

$$\vec{r}_2' = \vec{r}_2 - \vec{R}_{cm}$$

Centre of Mass Frame Coordinates

Laboratory frame Coordinates

$$\vec{r}_1' = \vec{r}_1 - \vec{R}_{cm}$$

Transformation to Centre of Mass

More intuitive is

$$\begin{aligned} \vec{r}_1 &= \vec{r}_{cm} + \vec{r}_{1,cm} \\ &= \vec{r}_{cm} + \vec{r}_1' \end{aligned}$$

$$\vec{r}_1' = \vec{r}_1 - \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$= \frac{m_2 \vec{r}_1 - m_2 \vec{r}_2}{m_1 + m_2} = \frac{m_2 (\vec{r}_1 - \vec{r}_2)}{(m_1 + m_2)}$$

$$\vec{r}_2' = \frac{m_1 (\vec{r}_2 - \vec{r}_1)}{(m_1 + m_2)}$$

Let  $\vec{r}_1 - \vec{r}_2 = \vec{r}$

$$\Rightarrow \vec{r}_1' = \frac{m_2 \vec{r}}{m_1 + m_2} \quad \vec{r}_2' = -\frac{m_1 \vec{r}}{(m_1 + m_2)}$$

Let  $\mu = \left( \frac{m_1 m_2}{m_1 + m_2} \right)$

$$\Rightarrow \vec{r}_1' = \frac{\mu \vec{r}}{m_1}$$

$$\vec{r}_2' = -\frac{\mu \vec{r}}{m_2}$$

$$m_1 \vec{r}_1' = \mu \vec{r}$$

$$m_2 \vec{r}_2' = -\mu \vec{r}$$

$$\Rightarrow m_1 \vec{r}_1' + m_2 \vec{r}_2' = 0$$

Hence, collinear proved.

In centre of mass frame,

$$\sum m_i \vec{r}'_i = 0$$

~~momentum of mass about centre of mass is 0~~

$$\vec{r}'_i = \vec{r}_i - \vec{r}_{CM}$$

$$\vec{r}'_{CM} = \vec{r}_{CM} - \vec{r}_{CM} = 0$$

★

In lab frame,

$$\vec{r}_{CM} = \frac{\sum \vec{r}_i m_i}{\sum m_i} \quad \text{--- (1)}$$

$$\vec{P}_{CM} = \sum P_i \quad \text{--- (3)}$$

$v_{CM}$  is constant if  $F_{ext} = 0$

$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{\sum m_i \vec{u}_i}{\sum m_i} \quad \text{--- (2)}$$

★

In COM frame,

ie.

$\vec{r}'_{CM} = 0$	--- (1)'
$\vec{v}'_{CM} = 0$	--- (2)'
$M \vec{v}'_{CM} = 0$	--- (3)'

If no external net force acts, we use this frame eg. collisions

Zero Momentum Frame



✓ All collisions are studied in COM frame reference.

$$\vec{r}_i' = \vec{r}_i - \vec{R}_{CM}$$

differentiating

$$\frac{d\vec{r}_i'}{dt} = \frac{d\vec{r}_i}{dt} - \frac{d\vec{R}_{CM}}{dt}$$

$$\vec{u}_i' = \vec{u}_i - \vec{u}_{CM}$$

COM frame

lab frame

$$\vec{u}_{CM} = \frac{\sum m_i \vec{u}_i'}{\sum m_i} = 0$$

(can be proved  
as we did for  
collinearity)

$$\vec{a}_{CM} = \frac{d\vec{u}_{CM}}{dt} = \frac{\sum m_i \left(\frac{d\vec{u}_i'}{dt}\right)}{\sum m_i} = \frac{\sum \frac{d}{dt}(m_i \vec{u}_i')}{\sum m_i}$$

$$M \vec{a}_{CM} = \sum \frac{d(\vec{p}_i)}{dt} = \frac{d}{dt} (\sum P_i)$$

$$\vec{F}_{ext} = \frac{d}{dt} (P_1 + P_2 + \dots + P_n)$$

$$\text{If } \vec{F}_{\text{ext}} = 0$$

$$\Rightarrow P_1 + P_2 + \dots + P_n = \text{const}$$

or

$$P_{\text{CM}} = \text{const.}$$

$$\vec{P}_{\text{CM}} = \vec{P}_1 + \vec{P}_2 + \dots + \vec{P}_n$$

$$\textcircled{1} \quad m_1 \vec{v}_1 + m_2 \vec{v}_2 + \dots + m_n \vec{v}_n = M \vec{v}_{\text{CM}}$$

$$\textcircled{2} \quad m_1 \vec{v}_1 + m_2 \vec{v}_2 + \dots + m_n \vec{v}_n = M \vec{v}_{\text{CM}}$$

$$\text{ie. } \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \dots + \vec{p}_n = \vec{p}_{\text{CM}}$$

$$\textcircled{3} \quad m_1 \vec{a}_1 + m_2 \vec{a}_2 + \dots + m_n \vec{a}_n = M \vec{a}_{\text{CM}}$$

$$\text{ie. } \vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n = \vec{F}_{\text{CM}}$$

उारे भय्ये थे  
ते पत ही होना  
चाहिए !!

Operator  $\vec{\nabla}$ 

$$\circ \vec{\nabla}^T = \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right)^T$$

$$\vec{\nabla} = \left[ \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right]$$

◦ 'del' is not a vector. It is a vector operator.

It differentiates the argument (scalar) and makes it a vector.

◦ 'del' mimics the behaviour of an ordinary vector in 3 unique ways:

1) Multiply by scalar  $\vec{A}_k \iff \vec{\nabla}^T$  or Gradient

2) Dot Product  $\vec{A} \cdot \vec{B} \iff \vec{\nabla} \cdot \vec{A}$  or Divergence

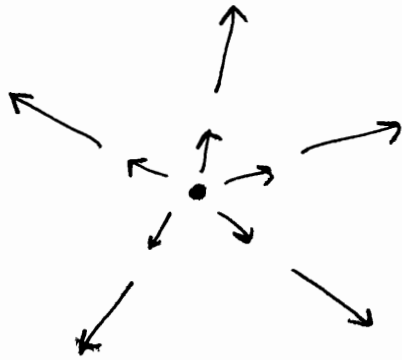
3) Cross Product  $\vec{A} \times \vec{B} \iff \vec{\nabla} \times \vec{A}$  or Curl

We have seen gradient. Let's see the other 2.

Divergence

$$\begin{aligned} \vec{\nabla} \cdot \vec{A} &= \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) \\ &= \left[ \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right] \end{aligned}$$

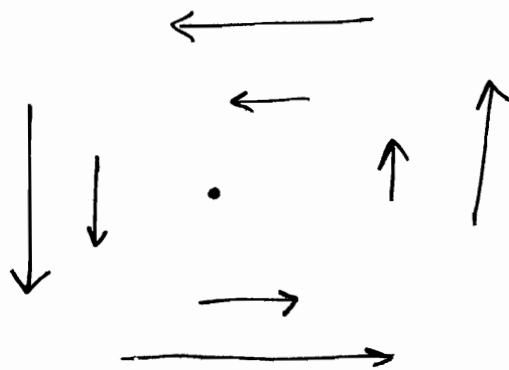
Geometrically, divergence of a vector  $\vec{A}$  is the measure of how much the vector  $\vec{A}$  spreads out from the observation point. HW



## Curl

$$\begin{aligned} \vec{\nabla} \times \vec{A} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\ &= \hat{i} \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) - \hat{j} \left( \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \\ &\quad + \hat{k} \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \end{aligned}$$

Geometrically, curl is a measure of how much the vector  $\vec{A}$  curls around the observation point.



## Integration by Parts

$$\int u dv = uv - \int v du$$

$$\text{ie. } \int \underset{\substack{\uparrow \\ \text{इसकी differentiation} \\ \text{आसान है !!}}}{f(x)} \underset{\substack{\uparrow \\ \text{इसकी integration} \\ \text{आसान है !!}}}{g'(x)} dx = f(x) g(x) - \int f'(x) g(x) dx$$

इसकी differentiation आसान है !!  
इसकी integration आसान है !!

$$\text{eg } \int x \cos x dx$$

$$= x \sin x - \int \sin x dx$$

$$= x \sin x + \cos x + C$$

## Condition for Coplanar Vectors

For 3 coplanar vectors, volume of the parallelepiped should be 0

$$\Rightarrow \vec{a} \cdot (\vec{b} \times \vec{c}) = 0 \Rightarrow \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = 0$$

## Cramer's Rule

$$a_1 x + b_1 y + c_1 z = d_1$$

$$a_2 x + b_2 y + c_2 z = d_2$$

$$a_3 x + b_3 y + c_3 z = d_3$$

$$\text{Now } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

$$x = \left( \frac{D_x}{D} \right), \quad y = \left( \frac{D_y}{D} \right), \quad z = \left( \frac{D_z}{D} \right)$$

○ if  $D \neq 0 \Rightarrow$  unique solution

○ if  $D = 0$ , then

if  $D_x, D_y, D_z$  all are 0  $\Rightarrow \infty$  solution

if  $D_x$  or  $D_y$  or  $D_z \neq 0 \Rightarrow$  no solution

★ We can deduce cosine law from dot product  $[\vec{A} + \vec{B} = -\vec{C}]$ . We can deduce sine law  $[\vec{A} + \vec{B} = -\vec{C}]$  from cross product.



★  $\vec{v} = \vec{\omega} \times \vec{r}$ , where  $\vec{r}$  can be measured from any point on the axis of rotation.

★ Limitation of 3<sup>rd</sup> law of Newton

3<sup>rd</sup> law says that the forces exerted by the two interacting bodies over each other are equal and opposite provided that they are both measured simultaneously. If the time taken by the interaction of two bodies is sufficiently larger than the time taken by the light signal to travel from one body to the other, this requirement is almost fulfilled for ordinary practical purposes. But this law does not hold good for particles of atomic dimensions.

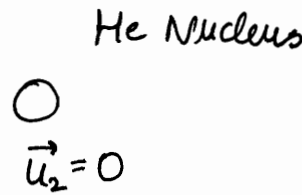
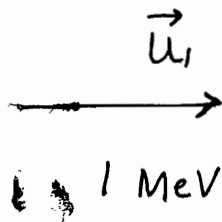
★ Galilean Transformations

Equations relating the 2 set of coordinates, both of which are of inertial frames of reference, are called Galilean Transformations. They are the particular case of more general LORENTZ TRANSFORMATIONS.

# MECHANICS (3)

17/11/11

Q3/Tut 1



$$\vec{p}'_1 = m_1 \vec{u}'_1$$

$$\vec{p}'_2 = m_2 \vec{u}'_2$$

$$\vec{u}'_1 = \vec{u}_1 - \vec{u}_{CM}$$

$$\vec{u}'_2 = \vec{u}_2 - \vec{u}_{CM}$$

$$\vec{u}_{CM} = \frac{m_1 \vec{u}_1 + m_2 \vec{u}_2}{m_1 + m_2}$$

$$= \frac{m_1}{m_1 + m_2} \vec{u}_1$$

$$\vec{u}'_1 = \frac{m_2}{m_1 + m_2} \cdot \vec{u}_1 = \frac{4}{5} \vec{u}_1$$

$$\vec{u}'_2 = -\vec{u}_{CM} = -\frac{1}{5} \vec{u}_1$$

★ CM parameters  
like  
 $\vec{r}_{CM} = \frac{\sum m_i \vec{r}_i}{\sum m_i}$   
 $\vec{u}_{CM} = \frac{\sum m_i \vec{u}_i}{\sum m_i}$   
can be directly  
used.....

$$E = \frac{hc}{\lambda} \quad \lambda = \frac{h}{mv}$$

$E = pc$  for a massless particle

$$E = \left( \frac{p^2}{2m} \right) \Leftarrow \text{Mass}$$

$$E_r = pc \Leftarrow \gamma \text{ ray}$$

Q1/Tut 1

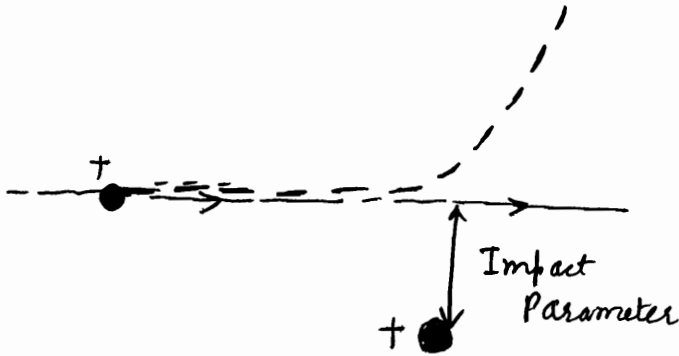
$$p_{recoil} + p_r = 0$$

$$p_r = -p_r$$

$$E_{recoil} = \frac{p_r^2}{2m} = \frac{p_r^2}{2m} = \frac{E_r^2}{2m c^2}$$

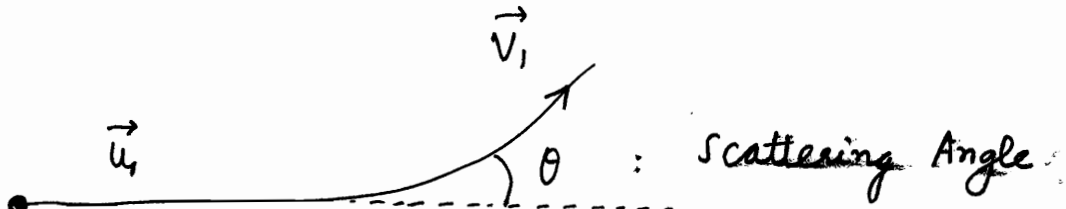
What we call 'Collision' in Mechanics (Classic) we call 'scattering' in Quantum Mechanics

Due to action of impulse, path of at least 1 particle changes, then collision is said to occur.

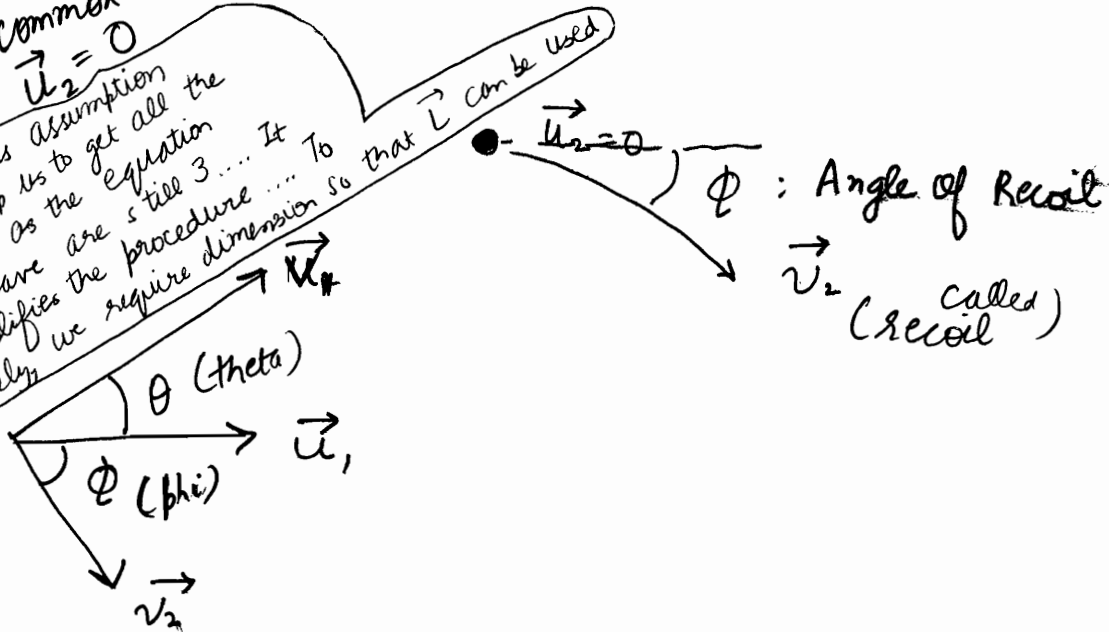


For collision, there is no necessity that particles will touch each other.

Impact Parameter: Perpendicular distance b/w incident particle and target particle in absence of any force.  
 (distance of closest approach if there were no interaction)



Most Common Case where  $\vec{u}_2 = 0$   
 Note that this assumption does not help us to get all the 4 parameters as the equation that we have are still 3.... It just simplifies the procedure.... To completely we require dimension so that  $\vec{L}$  can be used





Due to the impulse, change of momentum occurs.

$$\text{Impulse} = \int_i^f F \cdot dt = \int_i^f dp$$

$$= \vec{P}_f - \vec{P}_i$$

Collision can be of 2 types:

- 1) Elastic: <sup>Total</sup> kinetic Energy is conserved  
 i.e. no change of kinetic Energy to other energy.  
 i.e.  $\left(\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2\right) = \left(\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2\right)$
- 2) Inelastic: kinetic Energy is not conserved  
 i.e. there is a change of energy to other energy

Remember, total energy ~~is~~ is always conserved.

Momentum is of course conserved.

### Elastic Scattering

$$\begin{matrix} \text{(Photon)} & \text{(molecule)} \\ h\nu_i + E_1 & = E_1 + h\nu_s \end{matrix}$$

$$\Rightarrow \nu_i = \nu_s$$

Photon



—————  $E_2$

•••••  $E_1$

molecule

(Rayleigh Scattering)

### Inelastic Scattering

$$h\nu_i + E_1 = E_2 + h\nu_s$$

(Raman Scattering)

⊙ Collision is referred to as  $X + x = y + Y$

If some particles :- Collision  
before & after

If new particles :- transmutation  
afterwards

We know,

$$m_1 \vec{u}_1 + m_2 \vec{u}_2 = m_1 \vec{v}_1 + m_2 \vec{v}_2$$

$$\Rightarrow \frac{m_1 \vec{u}_1 + m_2 \vec{u}_2}{m_1 + m_2} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

$$\Rightarrow \vec{u}_{CM} = \vec{v}_{CM}$$

[As there is no external force, velocity of Centre of Mass remains const.]

## Elastic Collision

1) Momentum Conservation

$$2) \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

From these 2 equations, only for 1-d motion,

→ Momentum Conservation

$$\vec{v}_1 = \left( \frac{m_1 - m_2}{m_1 + m_2} \right) \vec{u}_1 + \left( \frac{2m_2}{m_1 + m_2} \right) \vec{u}_2$$

→ Coefficient of restitution for elastic collision = 1

$$\Rightarrow \vec{v}_2 - \vec{v}_1 = \vec{u}_1 - \vec{u}_2$$

$$\vec{v}_2 = \left( \frac{m_2 - m_1}{m_1 + m_2} \right) \vec{u}_2 + \left( \frac{2m_1}{m_1 + m_2} \right) \vec{u}_1$$

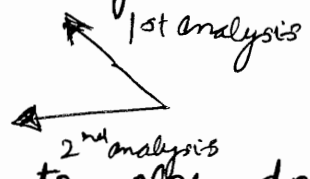
$$C.O.R. = \left( \frac{V.O.S}{V.O.A} \right)$$

[Final  
initial]

✓ If masses are same, the ~~old~~ velocity are exchanged.

Application

if  $u_2 = 0$



Moderator in nuclear reaction to slow down the neutron which are forced to collide with proton.

→ We can only solve Elastic Collision easily. Inelastic Collision requires other parameters to determine the state of motion after the collision.

Transformation

Taking most common case:  $u_2 = 0$

$$\vec{u}_{cm} = \frac{m_1 \vec{u}_1 + m_2 \vec{u}_2}{m_1 + m_2} = \frac{m_1 \vec{u}_1}{m_1 + m_2}$$

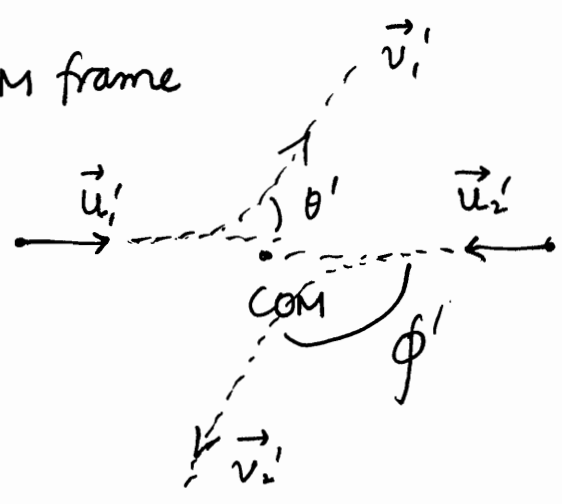
$$\vec{u}_1' = \vec{u}_1 - \vec{u}_{cm} = \frac{m_2}{m_1 + m_2} \vec{u}_1 = \frac{\mu}{m_1} \vec{u}_1$$

$$\vec{u}_2' = \vec{u}_2 - \vec{u}_{cm} = -\frac{m_1}{m_1 + m_2} \vec{u}_1 = -\frac{\mu}{m_2} \vec{u}_1$$

For simplicity, choose  $u_2 = 0$

direction from determined these equations

In COM frame



$$m_1 \vec{v}_1' + m_2 \vec{v}_2' = 0$$

Now 1 parameter is reduced as  $\phi' = \pi - \theta'$   
 This is because net momentum = 0.

$$\left. \begin{aligned} \theta &\rightarrow \theta' \\ \phi &\rightarrow \phi' \end{aligned} \right\} \text{ We want these transformations}$$

⊙ In COM Frame, applying energy & momentum conservation:

$$\vec{u}_2' = -\left(\frac{m_1}{m_2}\right) \vec{u}_1' \quad \left[ \text{Since Zero Momentum Frame} \right]$$

$$\vec{v}_2' = -\left(\frac{m_1}{m_2}\right) \vec{v}_1'$$

If elastic collision, [Note that KE is conserved w.r.t. all inertial frames]

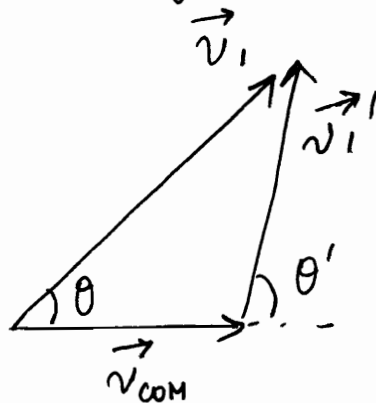
$$\frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$$

(Write  $u_2'$  in terms of  $u_1'$  &  $v_2'$  in terms of  $v_1'$ .... do not go back to laboratory frame  $\vec{u}_1$ )

$$\frac{1}{2} m_1 u_1'^2 + \frac{1}{2} \frac{m_1^2}{m_2} u_1'^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} \frac{m_1^2}{m_2} v_1'^2$$

$$\Rightarrow \boxed{u_1' = v_1'} \Rightarrow \boxed{u_2' = v_2'}$$

⊙ In elastic collision, in centre of mass frame, mag. of velocities do not change. Only direction of velocities ~~change~~ ~~change~~ change.



⊙  $\vec{v}_{CM} \parallel \vec{u}_1$   
hence this diagram could be made.

$$\vec{v}_1 = \vec{v}_1' + \vec{v}_{CM}$$

Resolving into components, we get transformation

$$v_{ix} = v_i' x + v_{cmx}$$

$$v_{iy} = v_i' y + v_{cm y}$$

⊛ directions are measured from direction of momentum of centre of mass.

$$v_i \cos \theta = v_i' \cos \theta' + v_{cm}$$

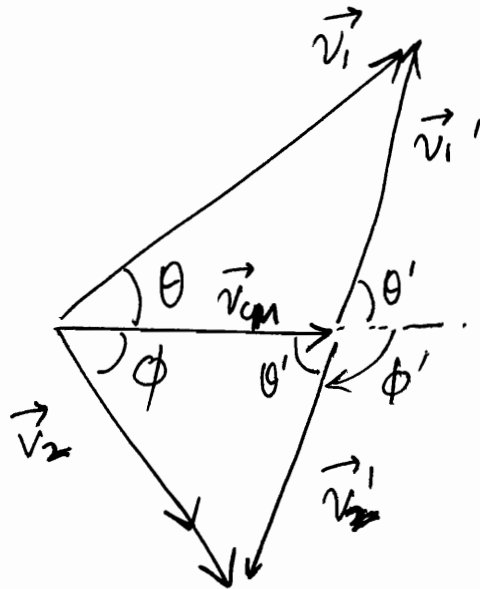
$$v_i \sin \theta = v_i' \sin \theta' + v_{cm} = v_i' \sin \theta'$$

dividing

$$\tan \theta = \frac{v_i' \sin \theta'}{v_i' \cos \theta' + v_{cm}} = \frac{\sin \theta'}{\cos \theta' + \left(\frac{v_{cm}}{v_i'}\right)}$$

$$\begin{aligned} \vec{v}_{cm} &= \vec{u}_{cm} \\ |\vec{v}_i'| &= |\vec{u}_i'| \\ \vec{u}_i' &= \frac{m_2}{m_1 + m_2} \vec{u}_1 \\ \vec{u}_{cm} &= \frac{m_1}{m_1 + m_2} \vec{u}_1 \end{aligned}$$

$$\tan \theta = \frac{\sin \theta'}{\cos \theta' + \left(\frac{m_1}{m_2}\right)}$$



$$\vec{v}_2 = \vec{v}_2' + \vec{v}_{cm}$$

$$\begin{aligned} v_{cm} - v_2' \cos \theta' &= v_2 \cos \phi \\ + v_2' \sin \theta' &= + v_2 \sin \phi \end{aligned}$$

$$\tan \phi = \frac{|\vec{v}_2'| \sin \theta'}{|\vec{v}_{cm}| - |\vec{v}_2'| \cos \theta'}$$

$$= \frac{|\sin \theta'|}{|\vec{v}_{cm}| - |\vec{v}_2'| \cos \theta'}$$

$$= \frac{|\sin \theta'|}{1 - \cos \theta'}$$

$$= \tan \left( \frac{\pi - \theta'}{2} \right)$$

$$v_{cm} = u_{cm} = \frac{m_1}{m_1 + m_2} u_1$$

$$v_2' = u_2' = -\frac{m_1}{m_1 + m_2} u_1$$

Note that only magnitudes are being considered.

$$\Rightarrow \boxed{\phi = \frac{\pi - \theta'}{2}} = \left[ \frac{\phi'}{2} \right]$$

All these results are for the standard case that ①  $\vec{u}_2 = 0$ .... THIS IS NORMALLY ASSUMED CASE.  
② ELASTIC COLLISION....

\* Coefficient of Restitution equations are used along normal.... hence if we use that equation, its wrong!!

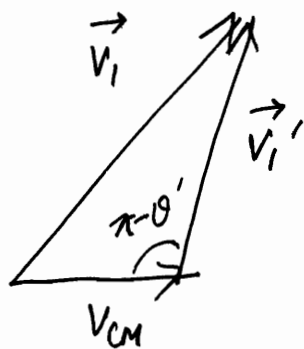
Q5 shows importance of assumptions.

we have to assume

- $u_2 = 0$
- elastic collision

$$u_{CM} = \frac{m_1 \vec{u}_1}{m_1 + m_2} = \vec{v}_{CM} = \frac{1}{1+A} \vec{u}_1$$

$$|\vec{v}_1'| = |\vec{u}_1'| = \frac{m_2}{m_1 + m_2} \vec{u}_1 = \frac{A}{1+A} \vec{u}_1$$



$$\cos(\pi - \theta') = \frac{v_1'^2 + v_{CM}^2 - v_1^2}{2 v_1' v_{CM}}$$

$$\frac{E'}{E} = \frac{v_1'^2}{u_1^2}$$

$$\Rightarrow v_1^2 = v_1'^2 + v_{CM}^2 + 2 v_1' v_{CM} \cos \theta'$$

$$= \frac{A^2}{(1+A)^2} u_1^2 + \frac{u_1^2}{(1+A)^2} + \frac{2A}{(1+A)^2} \cos \theta'$$

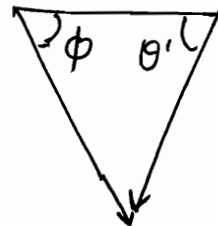
$$\Rightarrow \frac{v_1'^2}{u_1^2} = \frac{1 + A^2 + 2A \cos \theta'}{(1+A)^2}$$

\* Note the important thing that we are not given parameters like 'b', hence we have insufficient equation to get all variables,  $v_1, v_2, \theta, \phi$ . Hence we can at max find relation between variables in SCATTERING EXPERIMENTS !!!

o  $p_x, p_y, E$ : 3 equations while  $v_{1x}, v_{1y}, v_{2x}, v_{2y}$ : 4 variables

$$E_2 = \frac{1}{2} m_2 v_2^2$$

$$E_1 = \frac{1}{2} m_1 u_1^2$$



$$\frac{E_2}{E_1} = ?$$

Energy Transformation

$$E' = E - E_{cm}$$

Considering the conditions before collision,

$$E = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 u_1^2$$

$$E_{cm} = \frac{1}{2} (m_1 + m_2) u_{cm}^2 = \frac{1}{2} \frac{m_1^2 u_1^2}{(m_1 + m_2)}$$

$$= \frac{m_1}{m_1 + m_2} \cdot E$$

$$E' = \frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2$$

$$= \frac{1}{2} m_1 \frac{m_2^2}{(m_1 + m_2)^2} u_1^2$$

$$+ \frac{1}{2} m_2 \frac{m_1^2}{(m_1 + m_2)^2} u_1^2$$

$$= E \left[ \frac{m_2^2 + m_1 m_2}{(m_1 + m_2)^2} \right] = E \left[ \frac{m_2}{m_1 + m_2} \right]$$



$$E - E_{CM}$$

$$= E \left[ 1 - \frac{m_1}{m_1 + m_2} \right] = E \left[ \frac{m_2}{m_1 + m_2} \right]$$

⊛ In COM frame of reference, there is no restriction on  $\theta'$  while in lab frame,

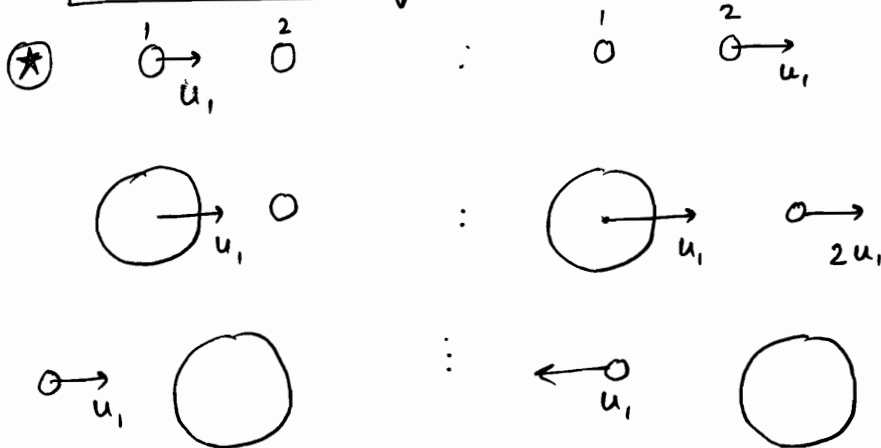
$$\tan \theta = \frac{\sin \theta'}{\cos \theta' + \left( \frac{m_1}{m_2} \right)}$$

This equation puts certain restrictions upon value of  $\theta$ .

eg. If  $m_1 > m_2 \Rightarrow \text{den} > 0 \Rightarrow \theta \neq (\pi/2)$

If  $m_1 < m_2 \Rightarrow$  all  $\theta$  are possible

3 special cases of 1-d collision



if  $m_1 = m_2$   
velocity exchanged

$m_1 > m_2$

$m_2 \gg m_1$

★ The term "flux" has two common usage in  
Physics :

- ① Flux as flow per unit area
- ② Flux as surface integral

(flow =  $\frac{dN}{dt}$ )  
in itself

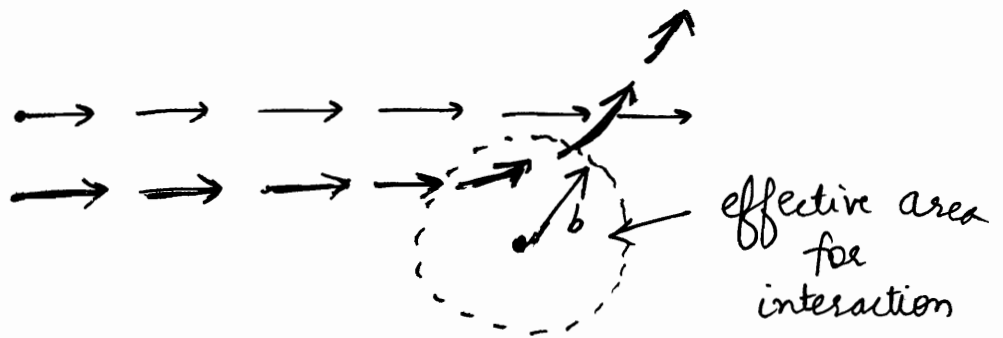
## Scattering Cross section

Every target particle offers an effective area for interaction.

Usually in scattering, we never fire single particle (we don't have such a source). We fire a beam of particles.

2 assumption in scattering:

- 1) same type of interaction b/w fired and target particle.
- 2) We consider, at a time, interaction b/w single particle and target. We neglect inter-particle interactions.



There is a Probability that scattering will occur.

• Scattering Cross section is the effective area presented by the target to the incident beam of particles. (I)

• It is also the fraction of  $\left( \frac{\text{particles scattered}}{\text{particles fired per unit area}} \right)$  The two components scattering cross section's definition

• Hence, it is the likelihood (probability) of particular scattering. (II)

# Scattering Cross Section

$$\sigma \text{ (units of area)} = \frac{\text{No. of particles scattering per unit time}}{\text{Incident Flux}}$$

$$\sigma = \left( \frac{N}{I} \right)$$

Incident Flux = Incident particle intensity  
 (I) = No. of particles fired per unit time per unit area.

$$\sigma = \pi b^2$$

~~Using 'flux' is wrong. The flux is obtained by dividing the number of particles per unit area.....~~

## Differential Scattering Cross Section

⊗ It is the additional number of particles scattered per unit increment in solid angle per unit flux incident.

$$\sigma(\Omega) = \left( \frac{dN}{d\Omega I} \right) = \left( \frac{d\sigma}{d\Omega} \right)$$

differential solid angle

It is the fraction of particles scattered per unit solid angle.

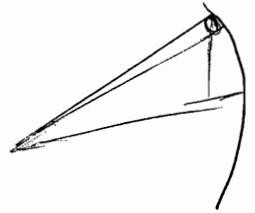
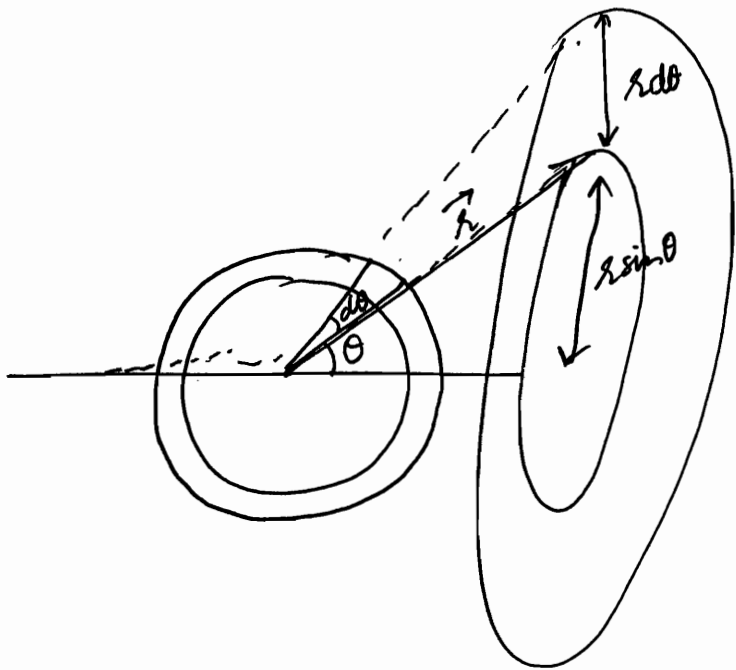
$$\sigma(\Omega) d\Omega = \frac{dN}{I}$$

We know,

$$d\Omega = \left( \frac{dA}{r^2} \right)$$

→ Solid Angle,  $\Omega$ , is the 2-d angle in 3-d space that an object subtends at a point

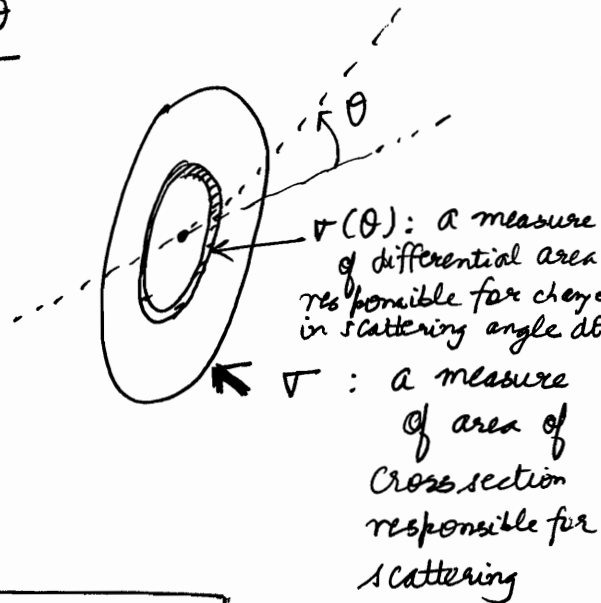
It is a measure of how large an object appears to an observer.



$\Omega$  : solid angle  
 $\theta$  : normal angle

$$d\Omega = \frac{2\pi r \cdot \sin\theta \cdot r d\theta}{r^2}$$

$$d\Omega = 2\pi \sin\theta d\theta$$



$$\int d\sigma = \int \sigma(\Omega) d\Omega$$

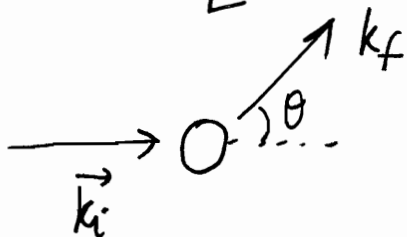
$$\sigma = \int_{\theta=0}^{\theta=\pi} \sigma(\theta) \cdot 2\pi \sin\theta \cdot d\theta$$

$$\frac{L \cdot N \cdot S}{I} \int dN = \frac{N}{I} = \sigma$$

$$\frac{R \cdot N \cdot S}{I} \int \sigma(\theta) \cdot d\Omega = \int \sigma(\theta) \cdot 2\pi \sin\theta d\theta$$

[scattering cross section] [differential scattering cross section]

Q4)



$$(\vec{k}_i - \vec{k}_f)^2 = (\vec{k}_i - \vec{k}_f) \cdot (\vec{k}_i - \vec{k}_f)$$

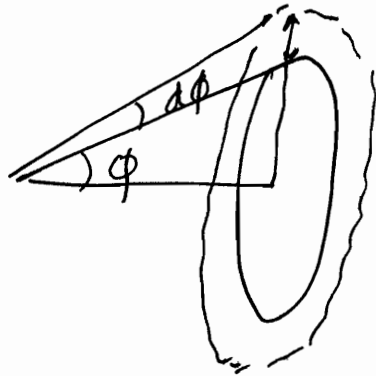
$$= 2k_i^2 (1 - \cos\theta)$$

$$\sigma(\theta) = A e^{-2B k_i^2} e^{2B k_i^2 \cos\theta}$$

$$\sigma = \int \sigma(\theta) 2\pi \sin\theta d\theta$$

Q4)  $\frac{dN}{dA} \propto \text{cosec}^4\left(\frac{\phi}{2}\right)$

$$\left(\frac{dN}{dA}\right) = N \text{cosec}^4\left(\frac{\phi}{2}\right)$$



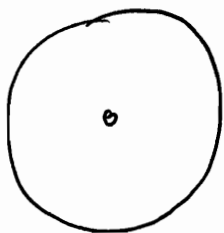
$$dA = 2\pi r^2 \sin\phi \, d\phi$$

$$\Rightarrow dN = N \text{cosec}^4\left(\frac{\phi}{2}\right) \cdot 2\pi r^2 \sin\phi \cdot d\phi$$

$$\frac{dN}{N} = \text{cosec}^4\left(\frac{\phi}{2}\right) \cdot 2\pi r^2 \sin\phi \cdot d\phi$$

$$\text{Fraction} = \int \text{cosec}^4\left(\frac{\phi}{2}\right) \cdot 2\pi r^2 \sin\phi \cdot d\phi$$

✓ Scattering Cross section is a function of target particle.



Neutron



Proton

# Transformation of <sup>Differential</sup> scattering Cross section

★ No. of particles scattered in lab frame or COM frame will remain same.

$$dN = I \sigma(\Omega) d\Omega$$

$$dN = I \sigma(\theta) 2\pi \sin\theta d\theta$$

$$dN' = I \sigma'(\theta') 2\pi \sin\theta' d\theta'$$

$$dN = dN'$$

$$\Rightarrow \sigma(\theta) \sin\theta d\theta = \sigma'(\theta') \sin\theta' d\theta'$$

$$\Rightarrow \sigma'(\theta') = \frac{\sigma(\theta) \sin\theta d\theta}{\sin\theta' d\theta'}$$

If elastic collision, we know

$$\tan\theta = \frac{\sin\theta'}{\cos\theta' + (m_1/m_2)}$$

Note that in both the cases we are talking about scattering and hence same  $\theta$  is being referred to.

eg. Proton-neutron elastic scattering

$$\tan\theta = \frac{\sin\theta'}{\cos\theta' + 1} = \frac{2 \sin(\theta'/2) \cos(\theta'/2)}{2 \cos^2(\theta'/2)} = \tan(\theta'/2)$$

$$\Rightarrow \theta = (\theta'/2) \Rightarrow \theta' = 2\theta$$

Put  $\theta' = 2\theta$

$$\Rightarrow \sigma'(\theta) = \frac{\sigma(\theta)}{2} \frac{\sin \theta d\theta}{\sin 2\theta d\theta} = \frac{\sigma(\theta)}{4 \cos \theta}$$

• If  $m_1 = m_2 \Rightarrow \boxed{\sigma'(\theta) = \sigma(\theta) / 4 \cos \theta}$

• If target particle is massive,  $\theta = \theta'$

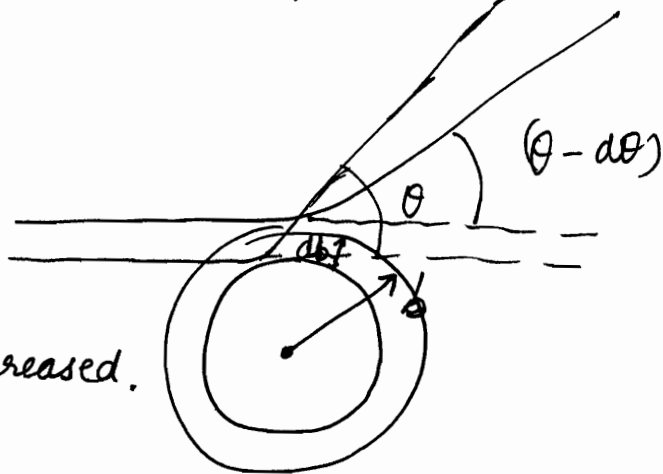
$$\boxed{\sigma'(\theta') = \sigma(\theta)}$$

Relation between  $b$  and  $\theta$

⊙ Note that for 1 particular collision of a particle out of the beam and target,  $b$  &  $\theta$  are specific for that particular collision.

If  $b$  is increased,  
 $\theta$  decreases

or If  $b$  is decreased,  $\theta$  increased.



$$dN = I \sigma(\theta) 2\pi \sin \theta d\theta$$

↑

Particles whose impact parameter is between  $[b]$  and  $[b - db]$

⊙ Note that  $2\pi b \cdot db$  is measure of  $\Delta\sigma$ . &  $\Delta\sigma = \left(\frac{\Delta N}{I}\right)$

$$= -I \cdot 2\pi b \cdot db \quad \left[ \begin{array}{l} \text{also increase in } \theta \text{ (i.e. } \oplus \text{ve } d\theta) \\ \text{means reduction in 'b'} \end{array} \right]$$

⊙ Hence to get  $\sigma(\theta)$ , I always require  $b$  as  $f(\theta)$

$$\sigma(\theta) = - \frac{b}{\sin \theta} \left( \frac{db}{d\theta} \right)$$

do not forget this negative sign

⊙  $\left(\frac{db}{d\theta}\right)$  is negative

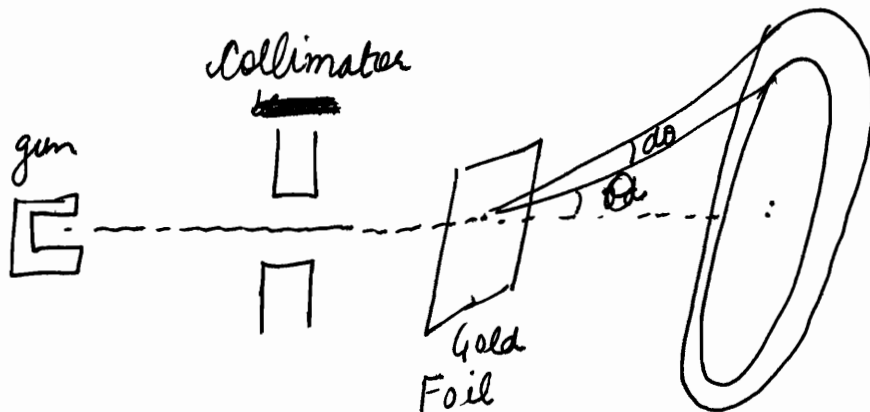
$\Rightarrow \sigma(\theta)$  is  $\oplus$ ve of course



# Rutherford Scattering

scattering of  $\alpha$ -particles by Gold Foil.

$$\sigma(\theta) = \left[ \frac{C^2}{16 E_\alpha^2} \right] \operatorname{cosec}^4 \left( \frac{\theta}{2} \right) \quad \therefore \text{everything in Centre of Mass Frame.}$$



⊙ Theoretical value does not match with actual figures. But no other formula comes as close.

We know

$$\frac{dn}{dr I} = \sigma(\theta) = -\frac{b}{\sin \theta} \left( \frac{db}{d\theta} \right)$$

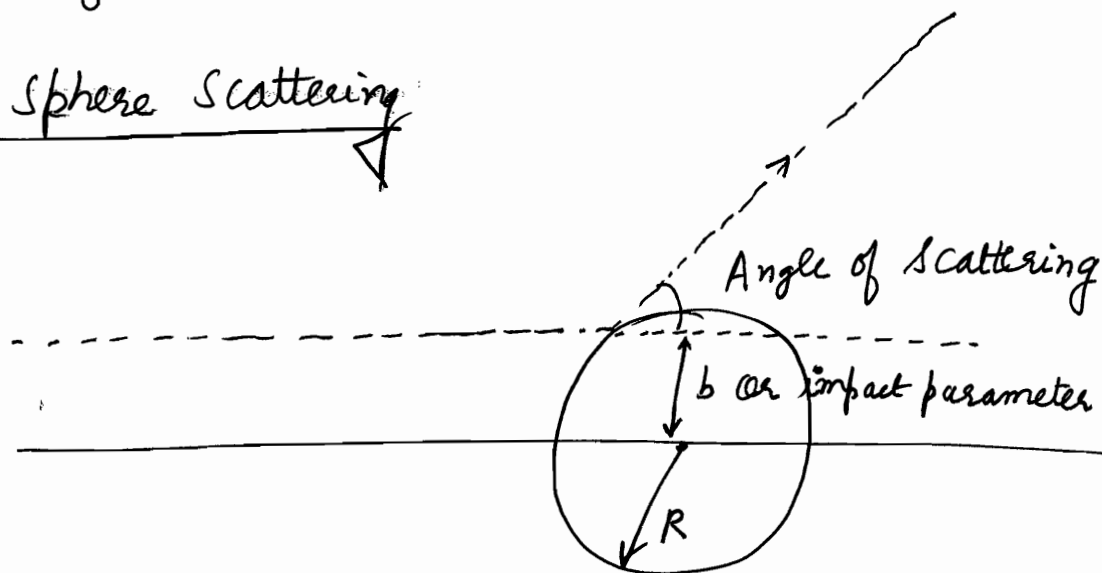
if  $b = \left( \frac{C}{2 E_\alpha} \right) \cot \left( \frac{\theta}{2} \right)$ , we get Rutherford formula

$$\sigma(\theta) = \frac{C}{2E_d} \frac{\cot(\theta/2)}{\sin \theta} \cdot \frac{C}{2E_d} [\operatorname{cosec}^2(\theta/2)] \frac{1}{2}$$

$$\sigma(\theta) = \frac{1}{16} \frac{C^2}{E_d^2} \operatorname{cosec}^4\left(\frac{\theta}{2}\right)$$

$$\odot \sigma = \int_0^\pi \sigma(\theta) \cdot 2\pi \sin \theta \, d\theta = \infty \text{ for above } \sigma(\theta)$$

### Hard Sphere Scattering

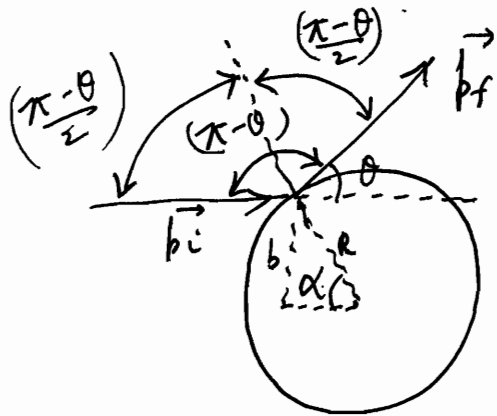


I am not interested in recoil

Angle of scattering =  $\angle$  b/w direction of approach of 1<sup>st</sup> particle and  $\angle$  of final path after getting scattered of same 1<sup>st</sup> particle.

Assume elastic collision

(\*) In order to find out  $\sigma(\theta)$ , we need  $b$  as a function of  $\theta$



Since elastic collision, and target much heavy, Using Conservation of k.E.

$$\Rightarrow \frac{p_i^2}{2m} = \frac{p_f^2}{2m}$$

$$\Rightarrow p_i = p_f$$

$$b = R \sin\left(\frac{\pi - \theta}{2}\right)$$

$$\text{or } (R + r_1) \sin\left(\frac{\pi - \theta}{2}\right)$$

$$b = R \cos\left(\frac{\theta}{2}\right)$$

↑  
if dimension of particle considered

$$\sigma(\theta) = \frac{-b}{\sin\theta} \left(\frac{db}{d\theta}\right)$$

$$= -R \cos(\theta/2)$$

$$-R \sin\left(\frac{\theta}{2}\right) \cdot \frac{1}{2}$$

$$2 \sin(\theta/2) \cos(\theta/2)$$

$$\sigma(\theta) = \left[ \frac{R^2}{4} \right]$$

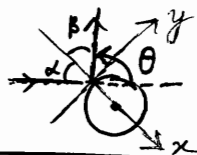
○ To Prove  $\alpha = \frac{\pi - \theta}{2}$

Target will provide impulse in direction normal to the surface, so that tangential momentum remains same

⇒

$$u_1 \sin \alpha = v_1 \sin \beta$$

$$u_1 = v_1 \Rightarrow \alpha = \beta$$



○ We can very easily see that this is the actual cross section provided by hard sphere.

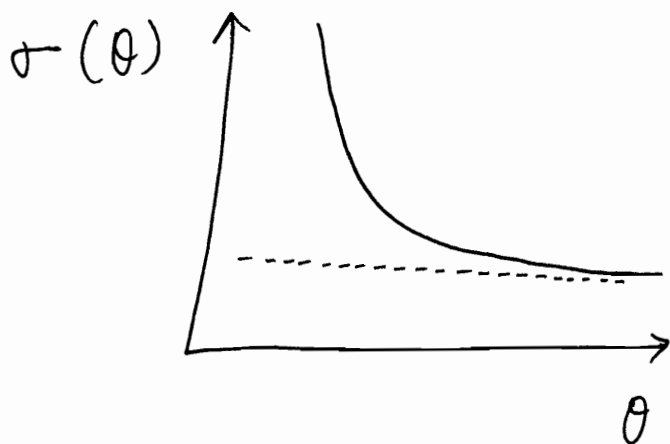
$$\sigma = \int_0^\pi \sigma(\theta) \cdot 2\pi \sin\theta \, d\theta$$

$$\sigma = \frac{\pi R^2}{2} \int_0^\pi \sin\theta \, d\theta = \pi R^2$$

$$\sigma = \pi R^2$$



$$\sigma = \pi [r_1 + r_2]^2 \quad (\text{general})$$

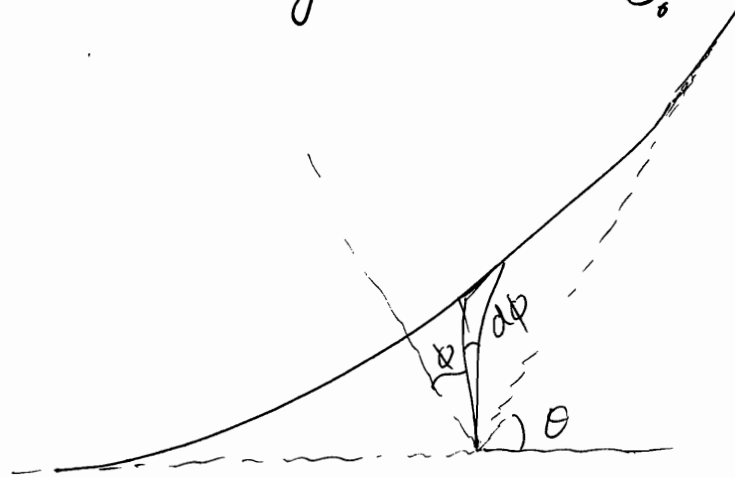


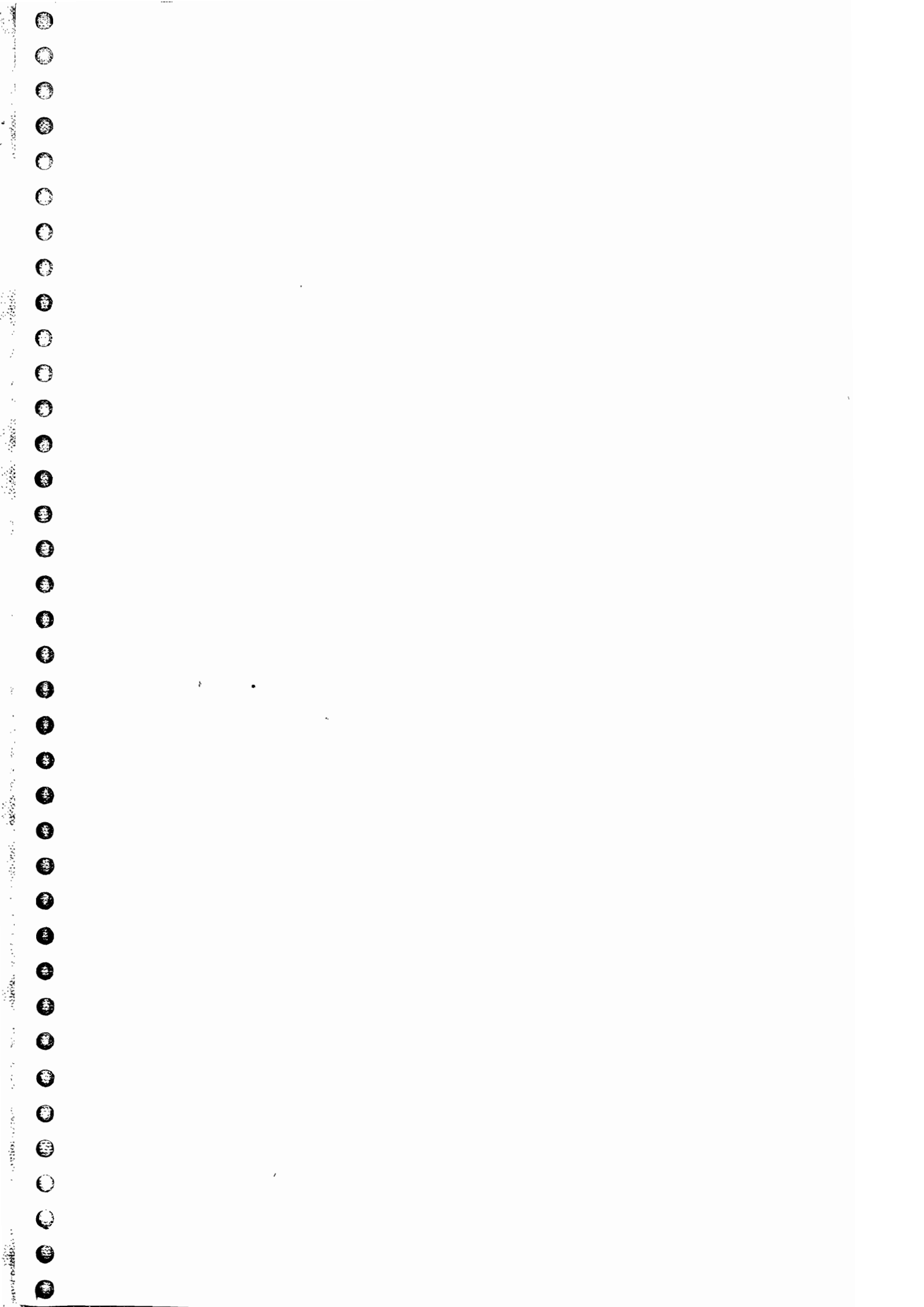
Rutherford  
Scattering

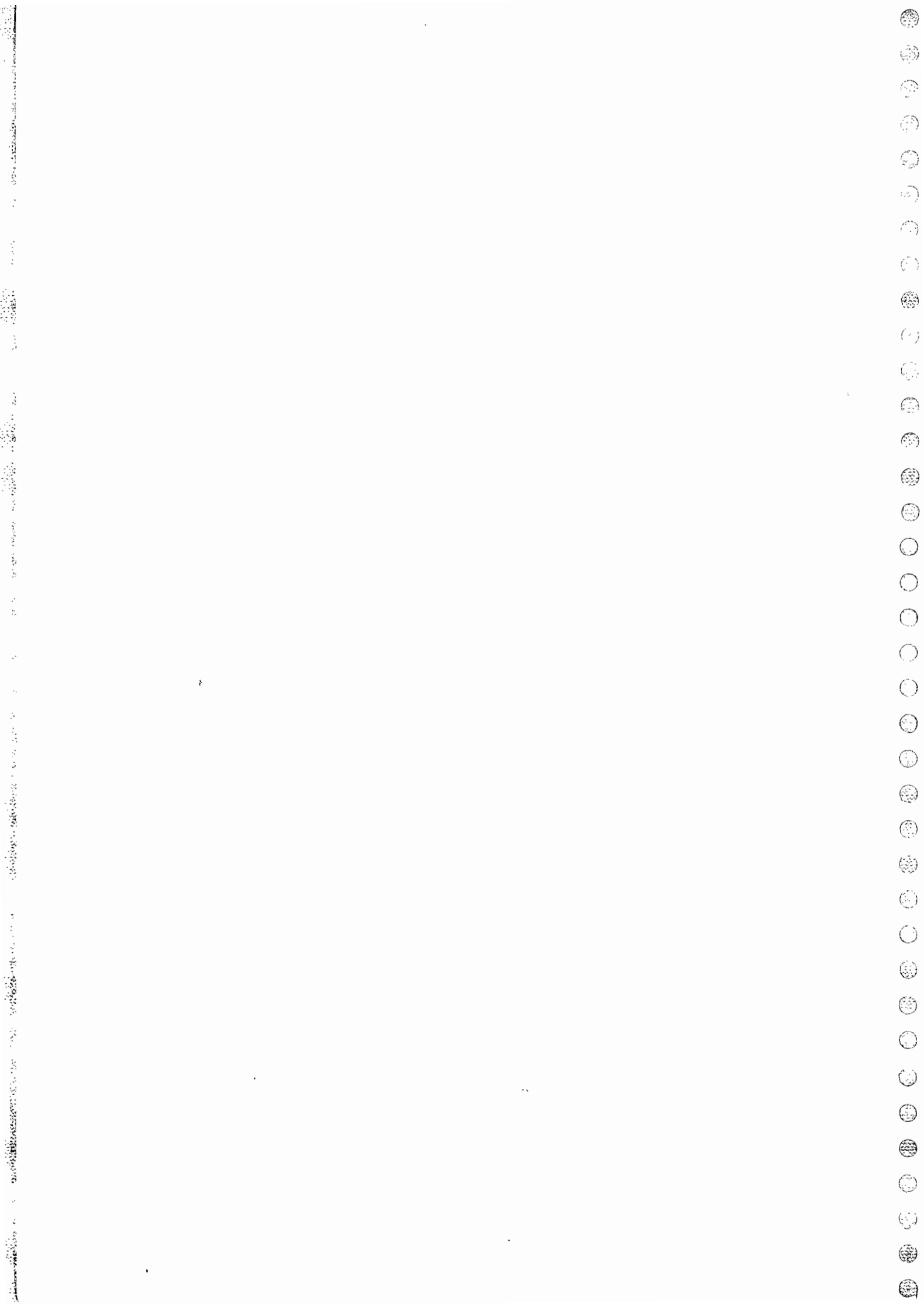
$$\sigma(\theta) \propto \text{cosec}^4\left(\frac{\theta}{2}\right)$$

→ Therefore the general steps are,

- ① From geometry & conservation laws, get  $b$  as function of  $\theta$
- ② From  $b$ , get  $\sigma(\theta)$  as  $-\frac{b}{\sin\theta} \left(\frac{db}{d\theta}\right)$
- ③ From  $\sigma(\theta)$ , get  $\sigma$  as  $\int_0^\pi \sigma(\theta) \cdot 2\pi \sin\theta \, d\theta$



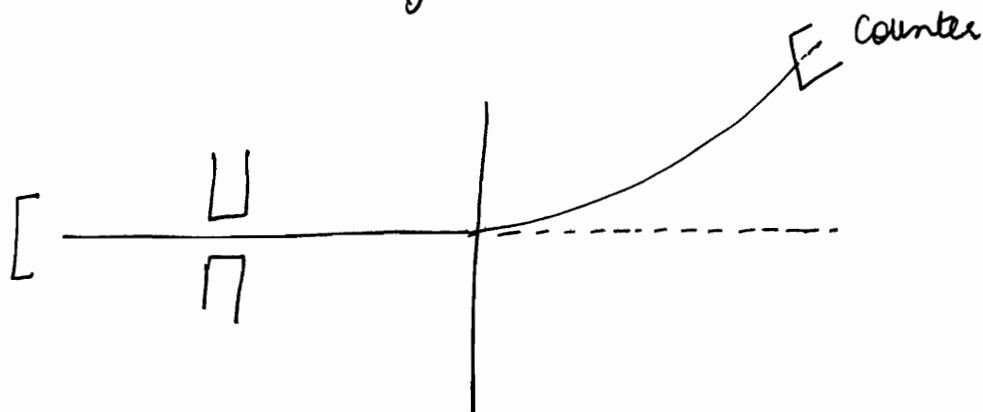




# MECHANICS (5)

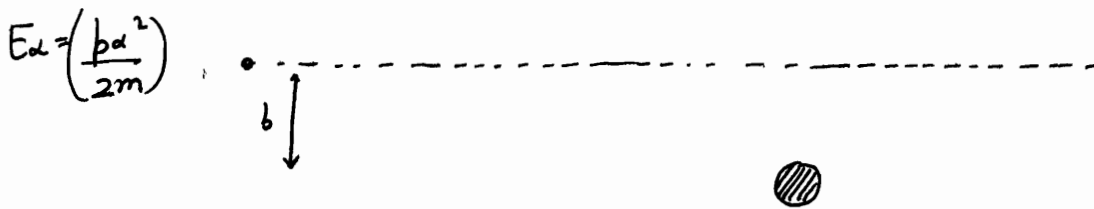
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## Rutherford Scattering (1912)



$$E_{\alpha} = \frac{1}{2} m u_{\alpha}^2 = \left( \frac{p_{\alpha}^2}{2m} \right)$$

(This is known as  $\alpha$ -gun)  
(fires at certain energy)



### Assumption involved in Rutherford Scattering:

- ✓ (1) Gold Foil Nucleus is very massive and hence remains at rest throughout the collision.
- ✓ (2) Both the particles are point particles [  $\alpha$ -particle and Gold Nucleus ]  
(So do not make sphere).
- ✓ (3) Collision is elastic between single incident particle and single target particle.
- ✓ (4) Scattering is due to  $\left( \frac{1}{r^2} \right)$  Coulumb Force.

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \left[ \frac{z_1 z_2 e^2}{r^2} \right]$$

\*  $r$  is measured from bigger target particle i.e. Centre of Mass Frame.

⊙ In all Rutherford results, Parameters are in Centre of Mass frame.

$$\int \vec{F} dt = \vec{p}_f - \vec{p}_i$$

✓ Note that the range of this force is  $\infty$ . We have neglected the electrons. We have considered only repulsive force of nucleus but not attractive force of electrons. Hence screening effect of electrons is neglected.

✓ Since force has  $\infty$  range  $\Rightarrow \sigma$  comes out to be  $\infty$ . But actually force upto a distance. Hence  $\sigma$  is found to be finite experimentally.

Since collision is elastic,

$$(k \cdot E)_i = (k \cdot E)_f$$

$$\Rightarrow v_i^2 = v_f^2$$

$$\Rightarrow |\vec{p}_i| = |\vec{p}_f|$$

Since Central Force is acting,  $\Rightarrow L=0$

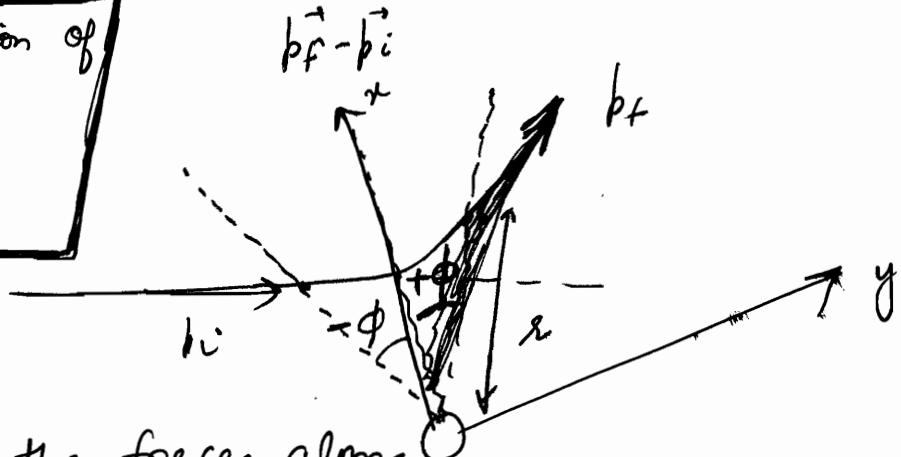
( $\because$  frame of reference is COM)  $\Rightarrow$  Angular Momentum is const.



$$\vec{L}_i = \vec{r} \times \vec{p}_i \Rightarrow L_i = p_i b$$

$\vec{F}_{net}$  is in direction of  $\vec{p}_f - \vec{p}_i$

If  $\hat{x}$  is the direction of  $(\vec{p}_f - \vec{p}_i)$  vector  $\Rightarrow$   
 $\int \vec{F}_x dt = (\vec{p}_f - \vec{p}_i)$

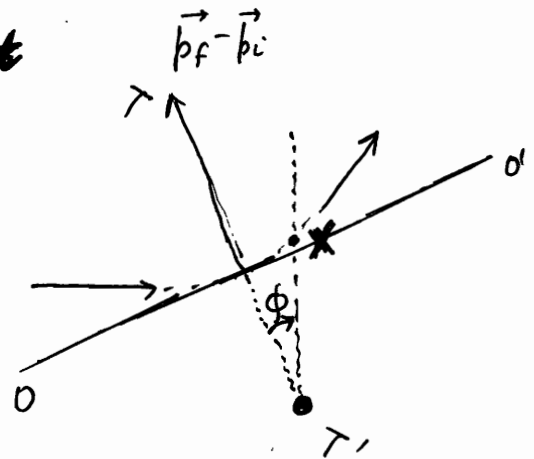
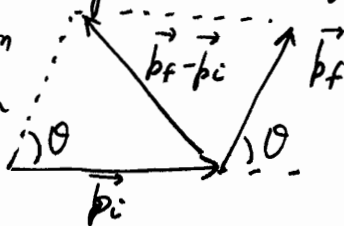


Let us consider the forces along the direction of  $(\vec{p}_f - \vec{p}_i) \cdot \hat{T}$ . Note that  $\int F dt$

$$L_f = m r^2 \left( \frac{d\phi}{dt} \right)$$

along the tangent  $OO'$  will come out to be zero.

At any point, applying equation of motion & conservation of angular momentum



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Note that we are measuring the angle from the normal

$$\frac{|\vec{p}_f - \vec{p}_i|}{\sin \theta} = \frac{|\vec{p}_f|}{\sin(\frac{\pi - \theta}{2})}$$

$$\Rightarrow \frac{\sqrt{2p^2 - 2p^2 \cos \theta}}{\sin \theta} = \frac{p}{\cos(\frac{\theta}{2})} \Rightarrow \frac{2p \sin(\frac{\theta}{2})}{\sin \theta} = \frac{p}{\cos(\frac{\theta}{2})}$$

$\Rightarrow$  equal....

$$\Rightarrow |\vec{p}_f - \vec{p}_i| = 2 p_i \sin\left(\frac{\theta}{2}\right)$$

$$\int F \cos \phi \frac{dt}{d\phi} \cdot d\phi = 2\pi i \sin\left(\frac{\theta}{2}\right) \quad \text{--- (1)}$$

$$\frac{dt}{d\phi} = \left(\frac{m\epsilon^2}{\pi i b}\right) \quad \text{--- (2)}$$

$$\Rightarrow \int_{\phi = \left(\frac{\pi-\theta}{2}\right)}^{\phi = -\left(\frac{\pi-\theta}{2}\right)} \frac{1}{4\pi\epsilon_0} \frac{Z_1 Z_2 e^2}{r^2} \cdot \cos \phi \frac{m\epsilon^2}{\pi i b} d\phi = 2\pi i \sin\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0} \frac{m}{\pi i b} \int_{\phi = -\left[\frac{\pi-\theta}{2}\right]}^{\phi = \left[\frac{\pi-\theta}{2}\right]} \cos \phi d\phi = 2\pi i \sin\left(\frac{\theta}{2}\right)$$

$$\Rightarrow b = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0} \frac{m}{\pi i^2} \cot \frac{\theta}{2}$$

$$\Rightarrow \boxed{b = \frac{C}{2E_\alpha} \cot\left(\frac{\theta}{2}\right)}$$

where  $C = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0}$

⊙ Note that all parameters in Rutherford Experiment are measured from Centre of Mass Frame.

So we need to convert the parameters to lab frame or vice versa.

$$\textcircled{\circ} \underline{\underline{E_{CM}}} = \frac{1}{2} (m_1 + m_2) u_{CM}^2 = \frac{1}{2} (m_1 + m_2) \left[ \frac{m_1 u_i}{(m_1 + m_2)} \right]^2$$

$$\textcircled{\circ} \underline{\underline{E_{CM}}} = \frac{1}{2} (m_1 + m_2) \cdot \frac{m_1^2 u_i^2}{(m_1 + m_2)^2} = \left( \frac{m_1}{m_1 + m_2} \right) E_d$$

$$\textcircled{\circ} \tan \theta = \left( \frac{\sin \theta'}{\cos \theta' + \frac{m_1}{m_2}} \right)$$

Rutherford Scattering Cross section

$\Leftrightarrow$  Differential Scattering Cross section

⊙ We have to mention

① Parameters in COM frame

② Specify  $\perp$  at the end.

in order to have a complete answer.

$$L_f = m r^2 \left( \frac{d\phi}{dt} \right)$$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$= r \hat{r} \times m (\dot{r} \hat{r} + r \dot{\phi} \hat{\phi})$$

$$= r m r \dot{\phi} \hat{\phi}$$

$$= m r^2 \dot{\phi} \hat{n}$$

$$\Rightarrow \underline{\underline{L_f = m r^2 \left( \frac{d\phi}{dt} \right)}}$$

Taking Polar Coordinates



along  $\hat{\phi}$  and directed

10) Larger the fraction of energy lost, better suited the moderator is.

(refer D.S. Mathur)

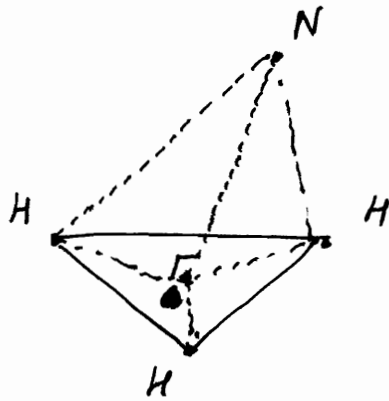
$$\text{loss of energy} = \frac{4 m_1 m_2}{(m_1 + m_2)^2}$$

That moderator would be used where loss of energy will be maximum.

it is maximum when  $m_1 = m_2$

But apart from mechanics, we have other parameters too. eg. ~~diff~~ scattering cross section of Proton is less than Neutron, absorption of neutrons by water.

15)



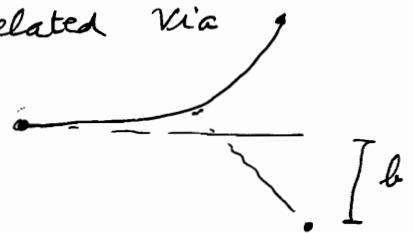
$$\frac{3}{17} C_N$$

$$\left( \frac{3 m_H C_N}{3 m_H + C_N} \right)$$

(\*) distance of closest approach and  $b$  are related via

$$(1) E_{\text{initial}} = \frac{1}{2} m v_a^2 + \frac{Ze^2}{4\pi\epsilon_0 r}$$

$$(2) m v_a r = \text{kin. } b$$



# Rocket Motion

① Rocket is a system of variable mass based upon Principle of Conservation of linear Momentum.

Escaping gases provide 'thrust' that propels the rocket forward.

## Equation of Motion

$$M \frac{d\vec{v}}{dt} = \vec{F}_{net} = \vec{F}_{wt} + \vec{F}_{thrust}$$

$$M \frac{d\vec{v}}{dt} = (-mg) + \left( + u \left( \frac{dm}{dt} \right) \right)$$

$$M \frac{d\vec{v}}{dt} = -mg + \vec{u} \left( \frac{dm}{dt} \right)$$

$$u = u_{exhaust}$$

Rocket Equation

This should be necessarily included in rocket motion question irrespective of whether asked or not

$\left( \frac{dm}{dt} \right)$  is ~~negative~~ positive,  $\vec{u}$  is negative

$\frac{dM}{dt}$  is negative

$$\frac{dv}{dt} = -g - \vec{u} \frac{1}{M} \left( \frac{dm}{dt} \right)$$

$$\int \frac{dv}{dt} \cdot dt = \int \left( -g - u \frac{1}{M} \left( \frac{dm}{dt} \right) \right) dt$$

Beyond Earth's Gravitation,  $g = 0$

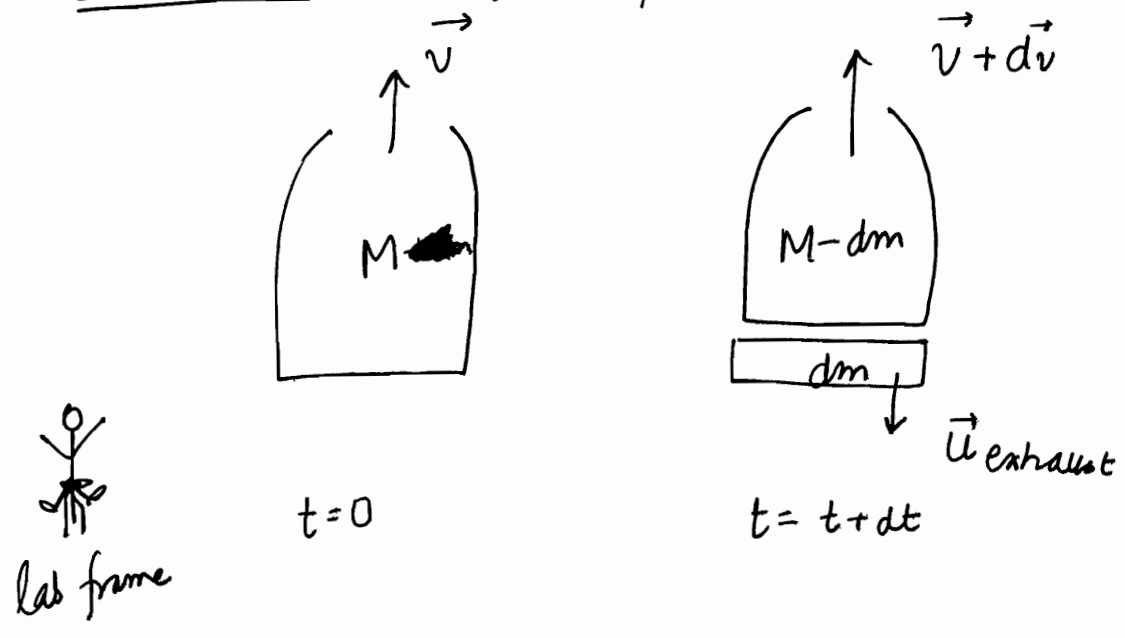
$$\Rightarrow \int_{v_0}^v dv = - \int_0^t g dt - u_{exhaust} \int_{M_0}^M \frac{dm}{M}$$

$$\Rightarrow v = v_0 - gt - u_{exhaust} \log \left( \frac{M}{M_0} \right)$$

$$\Rightarrow v = v_0 - gt + u_{\text{exhaust}} \ln \left( \frac{M_0}{M} \right)$$

No vectors involved, everything is  $\oplus$ ve.

derivation Refer last page for derivation



Exhaust velocity is always given in rocket's frame of reference.

$$\vec{u}_{\text{exhaust, earth}} = \vec{v}_{\text{rocket}} + \vec{u}_{\text{exhaust, rocket}}$$

$$\vec{u}_{\text{exhaust, earth}} = \vec{v} + d\vec{v} + \vec{u}_{\text{exhaust}}$$

$$\begin{aligned} \Delta \vec{p} &= \vec{p}_f - \vec{p}_i \\ &= (M - \Delta M) (\vec{v} + \Delta \vec{v}) + \Delta M (\vec{v} + \Delta \vec{v}) + \Delta M \vec{u}_{\text{ex}} \\ &\quad - M \vec{v} \end{aligned}$$

$$\Rightarrow \Delta \vec{p}_{\text{net}} = M \Delta \vec{v} + \Delta M \vec{u}_{\text{ex}}$$

$$\Rightarrow \frac{\Delta \vec{p}}{\Delta t} = M \left( \frac{\Delta \vec{v}}{\Delta t} \right) + \left( \frac{\Delta M}{\Delta t} \right) \cdot \vec{u}_{\text{exhaust}}$$

$$\lim_{t \rightarrow 0} \frac{\Delta \vec{p}}{\Delta t} = \frac{d\vec{p}}{dt} = M \left( \frac{d\vec{v}}{dt} \right) + \vec{u}_{\text{exhaust}} \left( \frac{dM}{dt} \right)$$

We know  $\frac{d\vec{p}}{dt} = \vec{F}_{\text{ext}} = m \left( \frac{d\vec{v}}{dt} \right) + \vec{u}_{\text{exhaust}} \left( \frac{dM}{dt} \right)$

$$m \left( \frac{d\vec{v}}{dt} \right) = \vec{F}_{\text{ext}} - \vec{u}_{\text{exhaust}} \left( \frac{dM}{dt} \right)$$

~~Note that here  $\frac{dM}{dt}$  is taken positive as  $dM$  is assumed to be the loss is weight~~

$$m \frac{d\vec{v}}{dt} = -g + \vec{u}_{\text{ext}} \ln \left( \frac{M_0}{M} \right)$$

$$\Rightarrow v = v_0 - gt + u_{\text{exhaust}} \ln \left( \frac{M_0}{M} \right)$$

Maximum  $u_{\text{exhaust}} = 4.5 \text{ km/sec}$  ✓

Maximum  $(M_0/M) = 4$  ✓

But the maximum velocity achieved is too less.

$$V_{\text{orbit (required)}} = 8 \text{ km/sec} \quad [\text{circular}]$$

$$V_{\text{escape (required)}} = 11.2 \text{ km/sec} \quad \begin{array}{l} [\text{elliptical}] \\ \vdots \\ \text{parabolic} \\ \text{hyperbolic} \end{array}$$

→ In order to increase the velocity, we have to add stages to the rocket.

In 1<sup>st</sup> stage

$$M_0 = 10,000 \text{ kg}$$
$$M = 2,500 \text{ kg}$$

$$M_{\text{fuel 1}} = 7500 \text{ kg}$$

In 2<sup>nd</sup> stage

$$M_0 = 2500 \text{ kg}$$
$$M = 500 \text{ kg}$$

$$M_{\text{fuel 2}} = 2000 \text{ kg}$$

⊙ 1<sup>st</sup> stage is largest in dimension & weight, and is used first and when its fuel is all burnt up and it has done its job, it get detached and is discarded, with 2<sup>nd</sup> stage taking over the task of producing further acceleration.

⊛ Apart from standard rocket eqn, we also have  $\beta$  parameter such that

$$M = M_0 (1 - \beta t)$$



# MECHANICS (6)

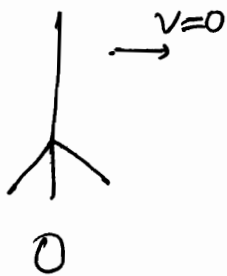
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## Rotating Frame of Reference

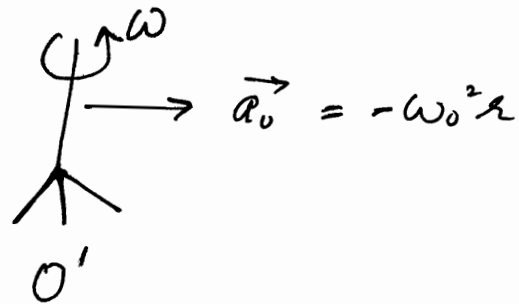
- expression for Coriolis & Centrifugal force
- deviation of a falling body
- Projectile on Earth
- deviation of  $g$  with centrifugal force

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2) \hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta}) \hat{\theta}$$

Newton's laws not valid in non inertial frame



inertial frame



non inertial frame

EOM for ~~from~~  $O'$ :  $O$ 's motion

$$F = -ma_0$$

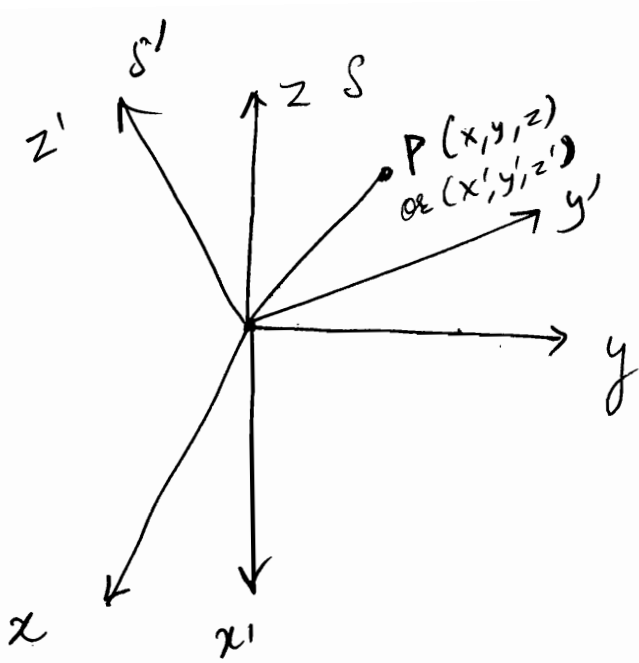
$\Rightarrow O'$  observes that a force  $[-ma_0]$  is acting on  $O$ . But no such force is acting.

This force is due to observation from non inertial frame. In order to make Newton's laws valid in non inertial frame of reference, we have to attach a pseudo force  $-ma_{frame}$  with the observed particle.

For above case  $\Rightarrow$

$$[\vec{F}_{real}]_{observed\ in\ non\ inertial\ frame} + \vec{F}_{pseudo} = [m\vec{a}]_{observed\ in\ non\ inertial\ frame}$$

$$F_{real} + [-ma_0] = m(-a_0) \Rightarrow F_{real} = 0$$



⊙ direction of angular velocity is perpendicular to the plane of rotation

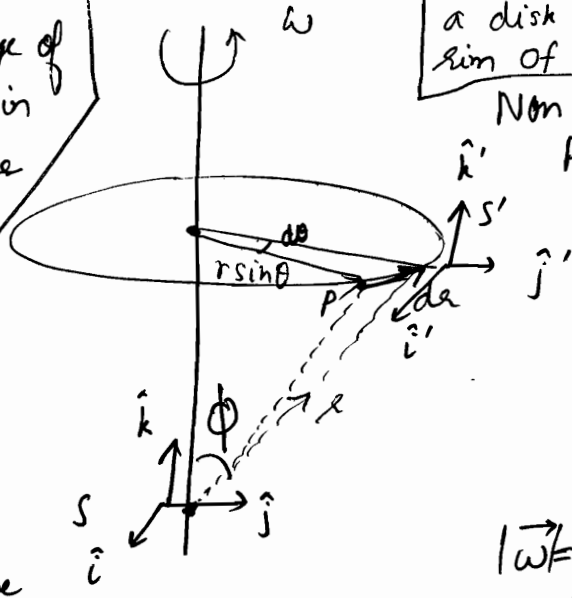
If I rotate frame of reference by  $|\vec{\omega}|$  angular velocity.  
(Body's frame)

$x, y, z$ : <sup>set of</sup> space coordinates

$x', y', z'$ : body set of coordinates [attached to the rotating body]

⊙ 1<sup>st</sup> step is to find the rate of change of a vector  $\vec{r}$  fixed in a rotating frame of reference

Consider a point P rotating with a disk at  $\vec{\omega}$  and located at rim of the disk.



Non Inertial Frame

$$\vec{v} = \left( \frac{d\vec{r}}{dt} \right)$$

$$d\theta = \frac{dr}{r \sin \phi}$$

$$|\vec{\omega}| = \left| \frac{d\theta}{dt} \right| = \frac{\left( \frac{dr}{dt} \right)}{r \sin \phi}$$

$$\left( \frac{d\vec{r}}{dt} \right) = \vec{\omega} \times \vec{r}$$

⇒ velocity measured from axis of rotation =  $[\vec{\omega} \times \vec{r}]$

$\hat{i}, \hat{j}, \hat{k}$  are fixed

$\hat{i}', \hat{j}', \hat{k}'$  are also rotating

Also  $\vec{r}$  measured from rotating frame = fixed = 0

⇒ For any ~~fixed~~ vector  $\vec{r}$  fixed in rotating frame, we have

For any rotating vector  $\vec{r}$

$$\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

if  $\vec{r}$  is fixed in rotating frame

general result

Next step is to find rate of change of a vector  $\vec{r}$  which is moving in a rotating frame of reference & described by unit vectors attached to frame of reference.

$$\frac{d\hat{i}'}{dt} = \vec{\omega} \times \hat{i}'$$

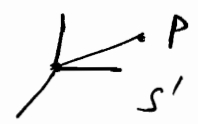
$$\frac{d\hat{j}'}{dt} = \vec{\omega} \times \hat{j}'$$

$$\frac{d\hat{k}'}{dt} = \vec{\omega} \times \hat{k}'$$

Take any Point P whose coordinates are

$(x, y, z)$  in S

$(x', y', z')$  in S'



Note that origin is a common point lying on the axis of rotation.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \text{ in frame S} \quad \text{--- (1)}$$

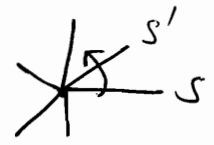
$$\vec{r} = x'\hat{i}' + y'\hat{j}' + z'\hat{k}' \text{ in frame S'} \quad \text{--- (2)}$$

If we consider common origin for S and S',  $\vec{r} = \vec{r}'$ .

$$\vec{v} = \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} + \frac{dz}{dt} \hat{k} \text{ in S}$$

$$\left(\frac{d\vec{r}}{dt}\right) = \vec{v}' = \frac{dx'}{dt} \hat{i}' + \frac{dy'}{dt} \hat{j}' + \frac{dz'}{dt} \hat{k}' \text{ in S'}$$

Notation



differentiating  $\vec{r}$

$$\left[ \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} + \frac{dz}{dt} \hat{k} \right]$$

$$= \left[ \frac{dx'}{dt} \hat{i} + x' \left( \frac{d\hat{i}'}{dt} \right) \right] + \left[ \frac{dy'}{dt} \hat{j}' + y' \left( \frac{d\hat{j}'}{dt} \right) \right] + \left[ \left( \frac{dz'}{dt} \right) \hat{k} + z' \left( \frac{d\hat{k}'}{dt} \right) \right]$$

$$= \vec{v}' + x' (\vec{\omega} \times \hat{i}') + y' (\vec{\omega} \times \hat{j}') + z' (\vec{\omega} \times \hat{k}')$$

$$= \vec{v}' + \vec{\omega} \times (x' \hat{i}' + y' \hat{j}' + z' \hat{k}')$$

$$\vec{v} = \vec{v}' + \vec{\omega} \times \vec{r}$$

$$\Rightarrow \frac{d}{dt} (\vec{r}) = \left[ \frac{d'}{dt} (\vec{r}) \right] + [\vec{\omega} \times (\vec{r})]$$

due to change of function w.r.t. rotating frame  
 change due to rotation of the frame  
 velocity measured in rotating frame

inertial frame

Transport Theorem

General Result :  $\frac{d}{dt} \vec{f} = \left( \frac{d\vec{f}}{dt} \right)_{s'} + \vec{\omega} \times \vec{f}$

$$\frac{d\vec{v}}{dt} = \frac{d'}{dt} (\vec{v}) + \vec{\omega} \times (\vec{v})$$

$$= \frac{d'}{dt} (\vec{v}' + \vec{\omega} \times \vec{r}) + \vec{\omega} \times (\vec{v}' + \vec{\omega} \times \vec{r})$$

∴ We have proved 2 things

- For a vector  $\vec{r}$  fixed in a frame  $S$ , moving at angular velocity  $\vec{\omega}$  w.r.t. a frame  $S'$ , we have  $\left( \frac{d\vec{r}}{dt} \right) = \vec{\omega} \times \vec{r}$
- For a vector  $\vec{f}$ , moving in  $S'$ ,  $\left( \frac{d\vec{f}}{dt} \right)_S = \left( \frac{d\vec{f}}{dt} \right)_{S'} + (\vec{\omega} \times \vec{f})$  ← Transport theorem

$$= \frac{d'v'}{dt} + \frac{d'}{dt} (\vec{\omega} \times \vec{r}) + \vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$a = a' + \underbrace{2\vec{\omega} \times \vec{v}'}_{\text{Contribution to Coriolis force}} + \underbrace{\left(\frac{d'\vec{\omega}}{dt}\right) \times \vec{r}}_{\text{Contribution to Euler Force}} + \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\text{Contribution to Centrifugal force}}$$

If uniform  $\omega$

$$\Rightarrow \vec{a}' = \vec{a} - 2(\vec{\omega} \times \vec{v}') - \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\Rightarrow m\vec{a}' = \underbrace{m\vec{a}}_{\text{true force}} - \underbrace{2m(\vec{\omega} \times \vec{v}') - m\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\text{Pseudo Components}}$$

remember,  $\underline{F' = F_{\text{real}} + F_{\text{pseudo}}}$  ; For this  $F'$ ,  $\underline{\frac{d'p'}{dt} = F'}$  is valid

$$\vec{F}_{\text{pseudo}} = -2m(\vec{\omega} \times \vec{v}') - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

⊛ do not forget this '2'

Coriolis Force

Centrifugal Force

If  $\vec{v}' = 0 \Rightarrow$  No Coriolis force only Centrifugal force

effect of centrifugal force:  
 $g' = g - \omega^2 R \cos^2 \theta$

- ⊙ Whenever a body moves w.r.t. rotating frame of reference, Coriolis force is acted upon the body.

$$\vec{F}_{\text{Coriolis}} = -2m (\vec{\omega} \times \vec{v}')$$

and

~~centrifugal~~

centrifugal force is acted upon the body

$$\vec{F}_{\text{centrifugal}} = -m \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

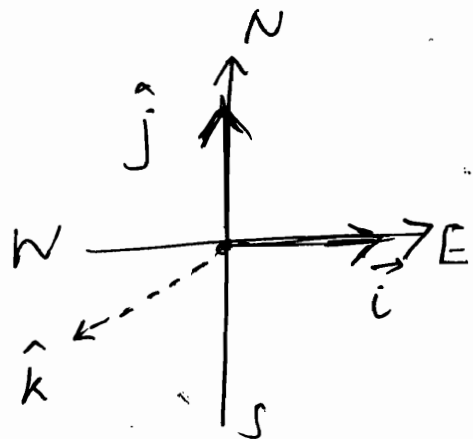
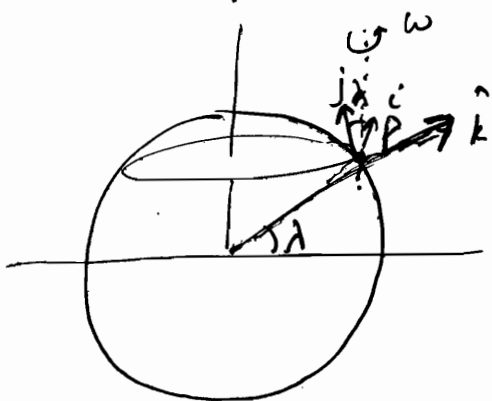
⊙  $\omega_{\text{earth}} = \frac{2\pi}{T} \approx 10^{-5}$  radians/second

- ⊙ For precise calculation, we need to attach these 2 forces for all motions seen w.r.t.

Earth.

All motions including flowing of currents, rivers, free falling body, throwing of projectile.

Free falling body on Earth



$$\vec{\omega} = \omega \cos \lambda \hat{j} + \omega \sin \lambda \hat{k}$$

$$\vec{v} = -v \hat{k}$$

$$\vec{F}_{\text{Coriolis}} = -2m (\vec{\omega} \times \vec{v})$$

$$\vec{\omega} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \omega \cos \lambda & \omega \sin \lambda \\ 0 & 0 & -v \end{vmatrix}$$

$$= -\hat{i} (+\omega v \cos \lambda)$$

$$\boxed{F_{\text{Coriolis}} = 2m \omega v \cos \lambda \hat{i}}$$

⊛ Note that we are adding Pseudo Force, hence we are measuring w.r.t. rotating frame i.e. we are considering the frame to be stationary. Hence, we do not have to worry about whether the point directly beneath will move to another point... Coriolis Force takes care of all such things.

towards East always  
← dependent upon  $|\vec{v}|$

Note that  $F_{\text{Coriolis}} \propto \vec{\omega} \times \vec{v}$   
 as soon as  $F_{\text{Coriolis}}$  act, after time  $\Delta t$ ,  $\vec{v}$  will have component along  $x$ -axis also and formula for  $F_{\text{Coriolis}}$  will become complicated.  
 But since  $F_{\text{Coriolis}}$  is so little than  $v_x \ll gt$  ( $\hat{k}$ )  
 Hence,  
 $v \approx -gt \hat{k}$

$$z = H \quad \dot{z} = 0 \quad \ddot{z} = 0$$

$$x = 0$$

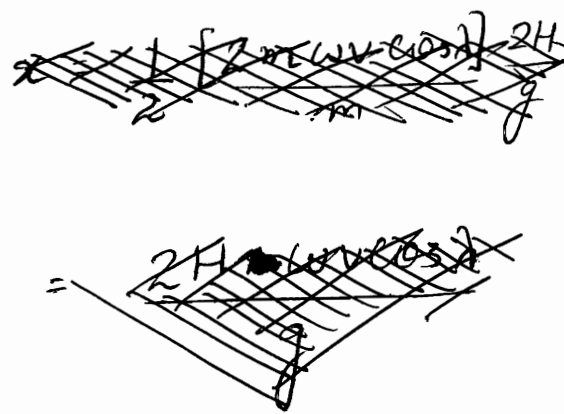
$$H = \frac{1}{2} g t^2$$

$$t = \sqrt{\frac{2H}{g}}$$

$$\vec{v} = -gt \hat{k}$$

$$v = gt$$

$$\frac{d^2 x}{dt^2} = 2 \omega g t \cos \lambda$$



$$x - 0 = \omega g t^2 \cos \lambda$$

$$\text{at } t=0 \quad x=0 \Rightarrow c_1=0$$

$$x = \frac{\omega g t^3}{3} \cos \lambda$$

$$\text{at } t=0, x=0 \Rightarrow c_2=0$$

$$x = \frac{\omega g}{3} \left[ \sqrt{\frac{2H}{g}} \right]^3 \cos \lambda$$

$\lambda$  is the latitude

⊙ 10 km से प्रियरे के वटा भए  
3 m से less का shift होगा !!

## Projectile Motion on earth's surface

$$\vec{v} = v_x \hat{i} + v_y \hat{j}$$

$\hat{i}$  : east  
 $\hat{j}$  : north

Same expression  
for  $\vec{\omega}$   
for both

$$\vec{\omega} = \omega \cos \lambda \hat{j} + \omega \sin \lambda \hat{k}$$

hemisphere, just

$\lambda$  will be negative  
in southern Hemisphere

F<sub>Coriolis</sub> =

$$-2m \vec{\omega} \times \vec{v}$$

$$\vec{\omega} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \omega \cos \lambda & \omega \sin \lambda \\ v_x & v_y & 0 \end{vmatrix}$$

$$= \hat{i} (-\omega v_y \sin \lambda) + \hat{j} (v_x \omega \sin \lambda) + \hat{k} (-v_x \omega \cos \lambda)$$



$$F_{\text{Coriolis}} = 2m\omega \sin \lambda (v_y \hat{i} - v_x \hat{j}) + 2m\omega v_x \cos \lambda \hat{k}$$

$$\vec{F}_{\text{horizontal}} = 2m\omega \sin \lambda [v_y \hat{i} - v_x \hat{j}]$$

$$\vec{F}_{\text{vertical}} = 2m\omega v_x \cos \lambda \hat{k}$$

From here we can see Ferrel's law

$$|\vec{F}_{\text{horizontal}}| = 2m\omega v \sin \lambda$$

Application of Coriolis force

$\Rightarrow \approx$  equator, Coriolis force = 0 } for motion on Earth's surface  
 $\Rightarrow \approx$  equator, no cyclones.

Most rivers flow from Poles to Equator  $\Rightarrow$  right bank of the rivers are more deeper due to Coriolis force.

Tut 2

(4)  $\vec{F} = \vec{F}_{\text{real}} + m(\vec{\omega} \times (\vec{\omega} \times \vec{r}))$

$$m'g' = m'g - m'\omega \times (\vec{\omega} \times \vec{r})$$

$$g' = 0 \Rightarrow g = \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$g' = g - \omega^2 R \cos^2 \lambda$$

1 component will also be there in  $\hat{j}$   
 $\omega^2 R \cos \lambda \sin \lambda$

$$\Rightarrow \omega n = \omega'$$

$$\frac{d}{dt} ( ) = \frac{d'}{dt} ( ) + \vec{\omega} \times ( )$$

Transform from rotating to inertial frame

o Value of  $\Gamma$  in Body frame of reference is known generally.

$$\frac{d\vec{J}}{dt} = \frac{d'}{dt} (\vec{J}') + \vec{\omega} \times (\vec{J}')$$

external Torque  $\vec{N} = \left[ \frac{d\vec{J}}{dt} \right] + (\vec{\omega} \times \vec{J})$

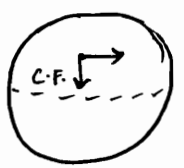
= torque in fixed frame  $\left( \frac{d\vec{J}}{dt} \right)$

Torque in rotating frame where Principal axis are fixed =  $I\vec{\omega}$  Refer Lecture 10

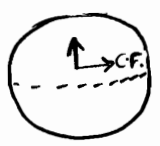
Euler's equation of motion in Rigid Body

NORTHERN HEMISPHERE :-  $\sin \lambda > 0$

(\*)



For a body moving parallel to latitude at velocity  $v_x \hat{i}$   
Coriolis Force  $\vec{F} = 2m\omega \sin \lambda [-v_x] \hat{j}$



For a body moving along PGL at  $\vec{v} = v_y \hat{j}$   
Coriolis Force  $\vec{F} = 2m\omega \sin \lambda [v_y] \hat{i}$

- (\*) Mainly remember that deflection due to vertical fall in terms of  $\cos \lambda$  (max. at equator)
- Deflection due to horizontal movement in terms of  $\sin \lambda$  (max. at poles)  
where  $\lambda$  : latitude
- Effect of centrifugal force in terms of  $\cos \lambda$  (max. at equator)

# Central Force Motion

- CONIC SECTION
- REDUCED MASS
- CENTRAL MOTION EQUATION
- GRAVITATIONAL MOTION EQUATION
- KEPLER'S LAWS AND PROOFS
- GAUSS LAW
- SELF ENERGY
- ANALYSIS OF SPHERE & SP

$$\vec{F} = F(r) \hat{r}$$

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times (F(r) \hat{r}) = 0$$

$$\Rightarrow \frac{d\vec{J}}{dt} = 0$$

$\Rightarrow \vec{J} = \text{constant of motion}$

$\Rightarrow [\vec{r} \times \vec{p}] = \text{constant of motion}$

$\Rightarrow$  direction of  $\vec{r} \times \vec{p}$  is fixed

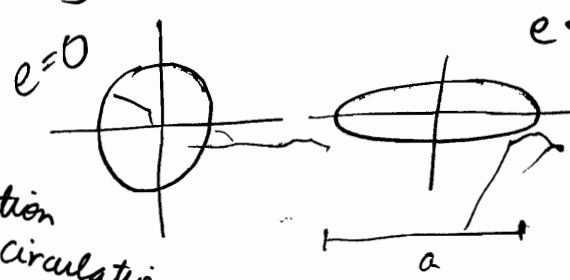
$\Rightarrow$  Motion is restricted to a plane

$\Rightarrow$  Planoar Motion

$\Rightarrow$  requires 2 coordinates, usually  $(r, \theta)$

Eccentricity = deviation from circular motion

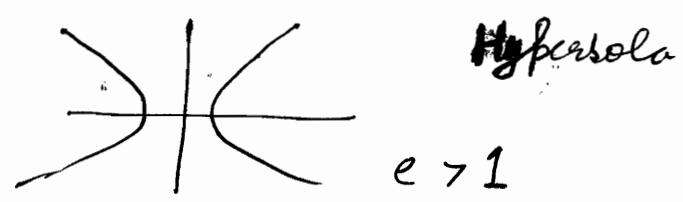
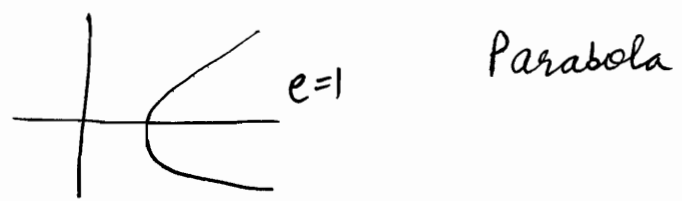
**Conic Sections**



$b$  closed curved motions in a plane

$\Rightarrow$  deviation from circular motion = 0

if  $a=b \Rightarrow$  circle  $\Rightarrow$  circle is particular case of ellipse

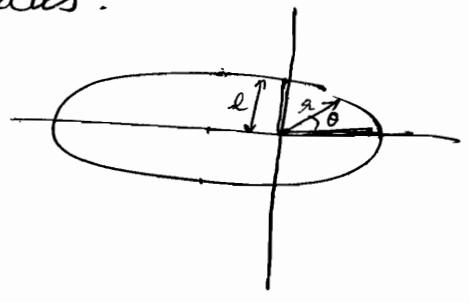


$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

general equation of conic section.

Its equivalent in  $(r, \theta)$  coordinates:

$$\frac{l}{r} = 1 + e \cos(\theta - \theta_0)$$



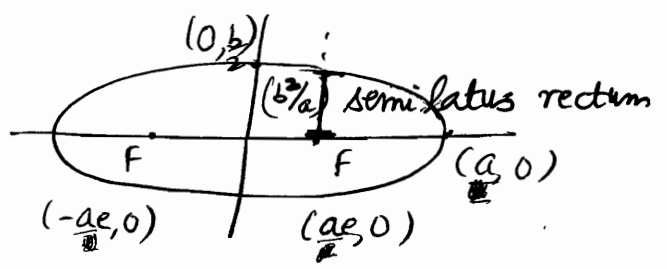
In ellipse, origin of axis is taken as Focus.

$l$ : semi latus rectum

( $1/2$ ) of length of the perpendicular drawn from focus parallel to directrix

$l = r$  for circle

$l = \left(\frac{b^2}{a}\right)$  for ellipse



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$b^2 = a^2(1 - e^2)$$

$$r = \frac{l}{1 + e \cos \theta}$$

$$\rightarrow r_{\min} = \left(\frac{l}{1+e}\right) = a(1-e)$$

$$\Rightarrow \frac{a^2 e^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow y^2 = b^2(1 - e^2) = \left(\frac{b^4}{a^2}\right)$$

$\Rightarrow y = \left(\frac{b^2}{a}\right)$   
 $\theta_0$  for slant of ellipse

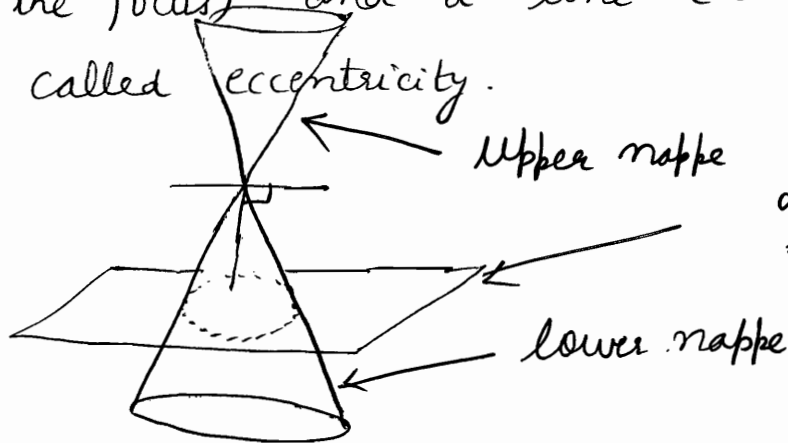
$$r_{\max} = \left(\frac{l}{1-e}\right) = a(1+e)$$

# HW

## Conic Section

Locus of points whose distances are in a constant ratio to a point (the focus) and a line (the directrix).

That ratio is called eccentricity.



Circle  
 $d = 0^\circ$   
 $\Rightarrow e = 0$   
 directrix @  $\infty$

$$e = \left[ \frac{\sin \alpha}{\sin \beta} \right]$$

$\alpha$ :  $\angle$  b/w plane & horizontal

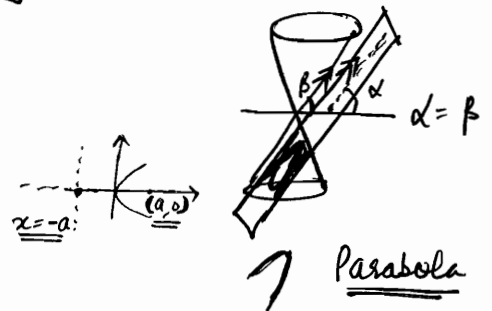
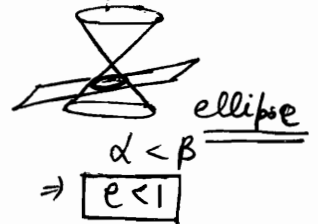
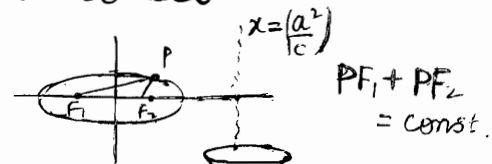
$\beta$ :  $\angle$  b/w cone & horizontal

For  $\beta = 90^\circ$

$e = 0$  if  $\alpha = 0^\circ$  [circle]

$0 < e < 1$  if  $0 < \alpha < 90$  [ellipse]

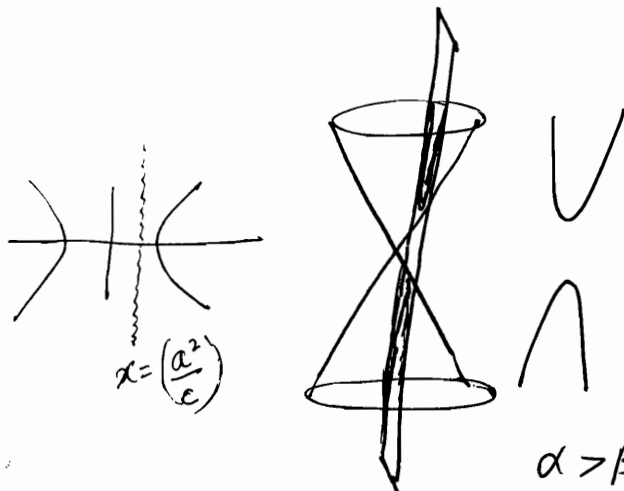
$e = 1$  if  $\alpha = 90^\circ$  [ ]



$$d = \beta$$

$$\Rightarrow \frac{\sin \alpha}{\sin \beta} = 1$$

$$\Rightarrow e = 1$$



Hyperbola

$\alpha > \beta$

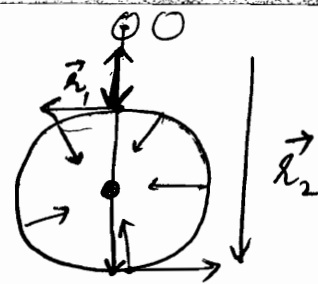
$$\Rightarrow e > 1$$

$$PF_1 - PF_2 = \text{Const.}$$

① Angular Momentum depends upon the frame of reference.

⇒ Central Force will be a central force depending upon whether  $\vec{F}_{ext} \parallel \vec{r}$ .

To simplify our calculations, we will take the point of reference s.t.  $\vec{F} \parallel \vec{r}$ .



a body performing circular motion under influence of gravitation force.

$$\vec{L}_1 = \vec{r}_1 \times m\vec{v} = mv r_1 (\hat{k})$$

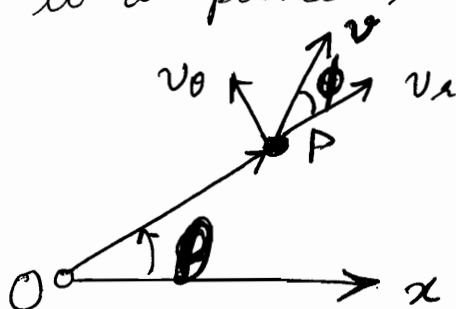
$$\vec{L}_2 = \vec{r}_2 \times m\vec{v} = mv r_2 (\hat{k})$$

⇒  $\vec{L}$  is changing

⇒  $L$  is not conserved.

## ① Angular Velocity [ $\vec{\omega}$ ]

Angular velocity of a particle is measured around or relative to a point, called the origin.



velocity of a particle has a component along the radius ( $v_r$ ) and a component perpendicular to the radius (cross-radial component, ( $v_\theta$ )). If there is no radial component, ~~the particle moves in a circle.~~ [note that, OP is only vector in path of particle P, hence result valid for all points] If there is no perpendicular component, particle moves along a straight line through the origin.

Radial Component produces no change in the direction of particle relative to origin, so it has no role in determining angular velocity. Rotation is

Completely produced by perpendicular motion around the origin, and hence angular velocity is completely determined by  $v_\theta$

⊙  $\omega = \left(\frac{d\theta}{dt}\right)$  .... definition in 2-d

We know  $v_\theta = r \left(\frac{d\theta}{dt}\right)$

In 2-d, direction of  $\vec{\omega}$  is not that important. It has a boolean direction vector only.

also  $v_\theta = v \sin \phi$

$\Rightarrow \boxed{\omega = \frac{v \sin \phi}{r}}$

Similarly in 3-d, we get

⊙  $\vec{\omega} = \frac{|\vec{v}| \sin \phi}{|\vec{r}|} \hat{u}$

In 3-d, direction of  $\vec{\omega}$  is of utmost importance

$\Rightarrow \boxed{\vec{\omega} = \frac{(\vec{r} \times \vec{v})}{|\vec{r}|^2}}$  .... in 3-d

where  $\vec{r}$  is the radius vector  
i.e.  $\vec{r} \cdot \vec{\omega} = 0$

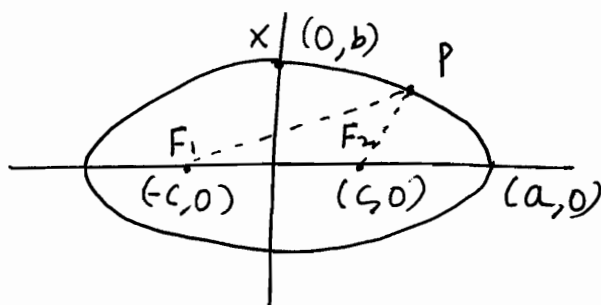
Note a perfect formula

Ellipse

$PF_1 + PF_2 = 2a$

~~$c = \sqrt{a^2 - b^2}$~~

$c = ae$



For point X,  $2a = 2\sqrt{b^2 + a^2e^2} \Rightarrow a^2(1-e^2) = b^2$

$\vec{v} = \vec{\omega} \times \vec{r}$  .... Perfect Formula

$\left(\frac{d\vec{r}}{dt}\right) = \left(\frac{d\vec{r}}{dt}\right)_R + (\vec{\omega} \times \vec{r}) = \vec{\omega} \times \vec{r}$

Body fixed in its own frame

- o Pseudo Forces enable us to determine whether or not a given frame is accelerated. Other forces go on rapidly decreasing with distance, fictitious forces have a considerable value proportional to acceleration of frame. If, therefore, we find a body or a particle, far away from other bodies, to be acted upon by an appreciable force  $\Rightarrow$  frame of reference is accelerated one....

MECHANICS  
was in  
COPY 2  
Now PTO To Last Page!!

### Foucault's Pendulum

It is a simple pendulum with massive bob carried by a very long wire & used to demonstrate rotation of Earth

$$l = 70 \text{ m} \quad m = 28 \text{ kg} \quad T = 17 \text{ sec}$$

(not matters)

Due to Coriolis force, the

plane of oscillation rotates due to rotation of Earth with

angular velocity  $\omega = \omega_0 \sin \phi$

$$\text{Precession velocity} = \omega_0 \sin \phi$$

$$T = \frac{2\pi}{\omega_0 \sin \phi} = \frac{(24 \text{ hr})}{\sin \phi}$$

### Newton's law of gravitation

Every particle of matter in the universe attracts every other particle with a force which is directly proportional to product of their masses & inversely proportional to square of the distance between them.

#### Field

Area around ~~with~~ a body within which its force of gravitational attraction is perceptible is called Gravitational Field. Intensity of the field is force per unit mass of a test mass



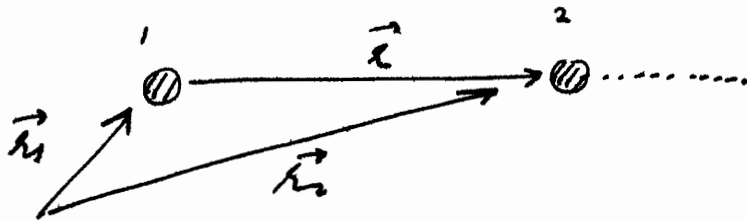
# MECHANICS (8)

23/11/11

- Euler Equation of Motion = Equivalent of Newton's Equation of Motion in rotation

i.e.  $T = \frac{d\vec{L}}{dt}$

Fields and Potentials



$$\vec{F}_{2,1} = - \frac{G m_1 m_2}{r^2} \hat{r}$$

- Every force has got its field. Gravitational Field is Influence of 1 mass (or charge in electromagnetics) in the space on a unit mass. (or other quantum in other types) of forces.

Force Field

$$\vec{E} = \left( \frac{\vec{F}}{m} \right)$$

Force

Quantum upon which force acts of  $\vec{E}$ .

⊙ For gravitational field, we can also use  $\vec{g}$  in place of  $\vec{E}$ .

Since  $\vec{F}$  is central force  $[|\vec{F}| \propto \frac{1}{r^2}]$

$\Rightarrow \vec{F}$  is conservative force

$$\Rightarrow \vec{F} = -\vec{\nabla} U$$

$$\Rightarrow \vec{E} = \frac{\vec{F}}{m} = -\vec{\nabla} \left( \frac{U}{m} \right)$$

$$V = \left( \frac{U}{m} \right)$$

$$\vec{F} \cdot d\vec{r} = -\nabla U \cdot d\vec{r} = -dU$$

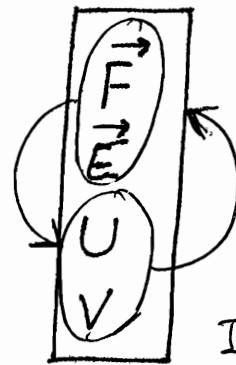
$$\Rightarrow |\vec{F}| = -\left(\frac{dU}{dr}\right)$$

$$\frac{F}{m} = E = -\frac{dU}{m dr}$$

Put  $\frac{dU}{m} = dV$

$$\Rightarrow E = -\left(\frac{dV}{dr}\right)$$

4 things we are interested in :



Important  
Conversions

$$\boxed{E = -\vec{\nabla}V}$$

$$\Rightarrow \boxed{dV = -\vec{E} \cdot d\vec{r}}$$

For a closed path,

$$\oint dV = -\oint \vec{E} \cdot d\vec{r} = 0$$

Reference at  $\infty$ ,  $V_{\infty} = 0 = V_{ref}$

$$\int_0^V dV = -\int_{ref}^r \vec{E} \cdot d\vec{r}$$

$\Rightarrow$

$$\boxed{V_r = -\int_{\infty}^r \vec{E} \cdot d\vec{r}}$$

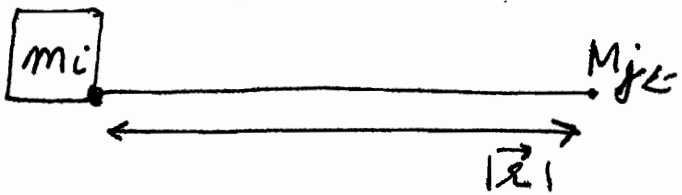
$$U = mV$$

$$\Rightarrow U_e = -m \int_{\infty}^{\vec{r}} \vec{E} \cdot d\vec{e}$$

$$\Rightarrow U_e = - \int_{\infty}^{\vec{r}} \vec{F} \cdot d\vec{e}$$

Here we are assuming that the mass  $m_i$  was placed before we started observing, thereby ignoring self energy, as now the work done has only gone into bringing  $m_j$  from  $\infty$  to  $\vec{r}$ .  
 ANYHOW SELF ENERGY FOR POINT CHARGES = 0

Consider mass  $m_i$  placed at origin and  $m_j$  be brought from  $\infty$  to  $\vec{r}$ :



Gravitational Potential Energy or Potential is always negative

$$\vec{F}_{j,i} = -G \frac{m_i m_j}{r^2} \hat{r}$$

Force on  $j$  due to  $i$

$$\vec{E}_j = \frac{\vec{F}}{m_j} = -G \frac{m_i}{r^2} \hat{r} = -G m_i \left( \frac{\vec{r}}{r^3} \right) = \nabla \left[ \frac{G m_i}{r} \right]$$

For a mass distribution,

$$d\vec{E}_j = -G \frac{dm_i}{r^2} \hat{r}$$

$$\left[ \frac{\vec{r}}{r^3} \right] = \nabla \left( \frac{1}{r} \right)$$

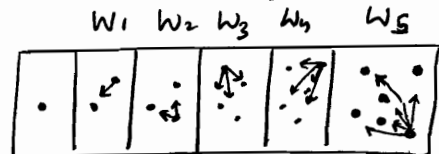
$$\begin{aligned} \nabla \left( \frac{1}{r} \right) &= \sum \hat{i} \frac{\partial}{\partial x} \left( \frac{1}{r} \right) = \sum \hat{i} \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{-1/2} \\ &= \sum \hat{i} \left( -\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} \cdot 2x \right) = -\frac{\vec{r}}{r^3} \end{aligned}$$

$$\vec{E} = \frac{-G m_i}{r^2} \hat{r} = -\vec{\nabla} V$$

$$V_j = -\frac{G m_i}{r_{ij}}$$

$$U = \sum_j m_j V_j = -G \frac{m_i m_j}{r_{ij}}$$

15 =  $\frac{30}{2}$  interactions



$$U = W_1 + W_2 + W_3 + W_4 + W_5$$

For a system of particles,

$$U = \frac{-1}{2} G \sum_{i,j} \left( \frac{m_i m_j}{r_{ij}} \right)$$

[ divide by half as terms will repeat ]

Newton's law of gravitation is an experimental law.

$$\begin{aligned} d\vec{E} &= -G \frac{dm}{r^2} \hat{r} \\ dV &= -\vec{E} \cdot d\vec{r} \\ dU &= m dV \end{aligned}$$

Alternate way to get  $\vec{E}$  is from Gauss's law

## ③ Methods to calculate $\vec{E}$ , $V$ , $U$

Note that if we get  $\vec{E}$ , others are known.

1) Basic law i.e. Newton's law  
or  
Coulumb's law

2) Gauss Law

3) Solution of Laplace and Poisson Equations

[we get  $V$ ,  $E$  is get from  $\vec{E} = -\vec{\nabla}V$ ]

4) Method of Images (in Electrostatics)

### Gauss Law

Every Force has got flux. Flux is the 'amount of field' in a unit area. Flux is the measure of flow of  $\vec{E}$ .

$$\text{Flux} = \int \vec{E} \cdot d\vec{s} = \text{No. of lines of force crossing some area}$$

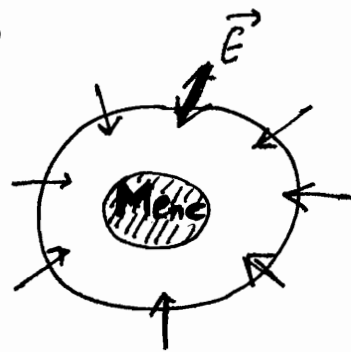
Flux of gravitational field is  $[-4\pi G]$  times the mass enclosed within the surface.

← This negative sign differentiates it from Electrical Field

Although law is valid for any type of surface but for proof, we choose a symmetric surface for easy integration.

$$\oint_S \vec{E} \cdot d\vec{s} = -4\pi G \text{ mass}$$

lets say mass is uniformly distributed,  
i.e.  $dm_{\text{enc}} = \rho dv$



$$\Rightarrow \oint_S \vec{E} \cdot d\vec{s} = -4\pi G \int_V \rho dv$$

integral form of gauss law

LHS

$$= \oint_S -\frac{GM \hat{r}}{r^2} \cdot d\vec{s}$$

$$= \oint_S -GM \hat{r} \left( \frac{d\vec{s}}{r^2} \right) = -GM \oint_S \frac{\hat{r} \cdot d\vec{s}}{r^2}$$

↑ solid angle

$$= -GM \oint_S d\Omega = -4\pi GM$$

### BASICS

Stoke's law: line integral to surface integral

$$\oint_L \vec{F} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{s}$$

Divergence law: surface integral to volume integral

$$\oint_S \vec{F} \cdot d\vec{s} = \int_V (\vec{\nabla} \cdot \vec{F}) dv$$

○ Curl represents rotation.  $\vec{\nabla} \times \vec{F}$

Irrotational field:  $\text{curl} = 0$

○ Divergence represents flow  $\vec{\nabla} \cdot \vec{F}$

○ Gradient represents slope.  $\vec{\nabla} T$

○ Curl of Gradient = 0

○ Laplace is divergence of gradient  $\Rightarrow$  scalar to a scalar

$$\oint \vec{E} \cdot d\vec{s} = -4\pi G \int_V \rho \, dv$$

$$\Rightarrow \int_V \vec{\nabla} \cdot \vec{E} \, dv = -4\pi G \int_V \rho \, dv$$

$$\Rightarrow \int_V (\vec{\nabla} \cdot \vec{E} + 4\pi G \rho) \, dv = 0$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{E} = -4\pi G \rho}$$

$\rightarrow$  Both the results are mathematical representation of Newton's law: converted by GAUSS.

$$\vec{E} = -\vec{\nabla} V$$

$$\Rightarrow \nabla^2 V = -\nabla \cdot \vec{E} = 4\pi G \rho$$

$$\Rightarrow \boxed{\nabla^2 V = +4\pi G \rho} \quad \text{Poisson Equation}$$

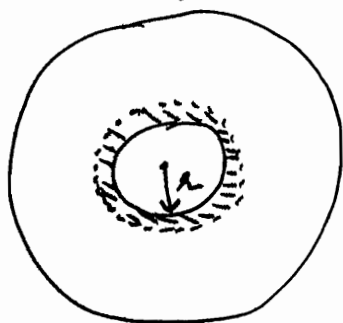
If no mass enclosed  $\Rightarrow \rho = 0 \Rightarrow \boxed{\nabla^2 V = 0}$  Laplace Equation

# Gravitational Self Energy

Amount of work done in assembling the system is gravitational energy.

Amount of work done in assembling a body by bringing infinitesimal masses from reference ( $\infty$ ) to the present location.

→ Let us take a sphere of mass  $M$  and uniform mass density  $\rho$ .



$$\rho = \frac{M}{\frac{4\pi r^3}{3}}$$

We need to find out Gravitational self energy  
let us take a core of radius  $r$ . Now around the core,  
Bringing  $dm$  shell from  $\infty$  around the core

$$dU = dm \cdot V$$

$$m = \frac{4}{3} \pi r^3 \rho$$

$$V_{\text{core}} = -\frac{Gm}{r} = -G \frac{4\pi r^2 \rho}{3}$$

$$dU = dm \cdot V$$

$$dU = -\frac{4G\pi\rho}{3} r^2 \cdot 4\pi r^2 dr$$

⊛ Surface Area of sphere  
 $= 4\pi r^2$

$$\Rightarrow \int dU = -\frac{4\pi}{3} G\rho \cdot 4\pi \int r^4 dr$$

$= r^2 \times \underline{4\pi} \leftarrow$   
Total Solid Angle



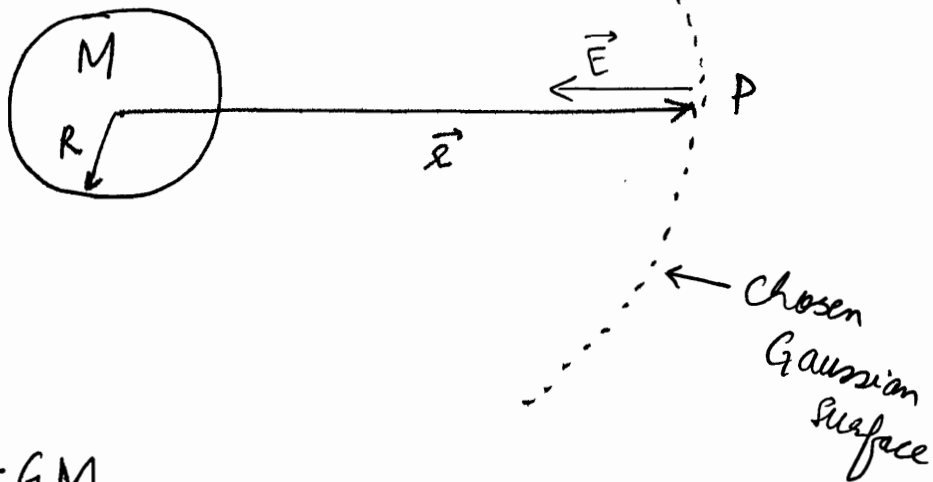
$$\Rightarrow \int dU = - \frac{3M}{8R^3} \cdot G \cdot \frac{3M}{R^3} \cdot \frac{R^5}{5}$$

$$\int dU = - \frac{3}{5} \left( \frac{GM^2}{R} \right)$$

$$\Rightarrow U = - \frac{3}{5} \left( \frac{GM^2}{R} \right)$$

do not forget the negative sign

### Applying Gauss in solid sphere



Point P can be

(i) Outside

$$\oint \vec{E} \cdot d\vec{s} = -4\pi GM$$

$$\Rightarrow - \oint E ds = -4\pi GM$$

$$\Rightarrow -E \oint ds = -4\pi GM$$

$$\Rightarrow E = \frac{4\pi GM}{4\pi r^2} = \frac{GM}{r^2}$$

$$\vec{E} = - \frac{GM}{r^2} \hat{r}$$

[ assumed  $\vec{E} = -E \hat{r}$  ]

$$V = - \int \vec{E} \cdot d\vec{r} = GM \int \frac{\hat{r} \cdot d\vec{r}}{r^2} = GM \int \frac{dr}{r^2} = - \left( \frac{GM}{r} \right)$$

(ii) At the surface [just put  $\vec{r}=R$ ]

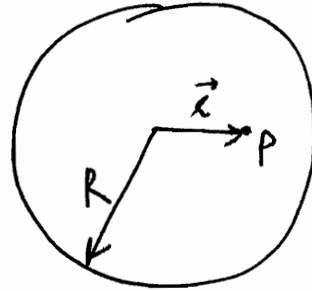
$$\vec{E} = -\frac{GM}{R^2} \hat{r}$$

$$V = -\left(\frac{GM}{R}\right)$$

(iii) Inside the sphere

$$\vec{E} \cdot 4\pi r^2 = -\frac{4\pi GM}{R^3} r^3$$

$$4\pi \vec{E} = -\frac{4\pi GM}{R^3} r^3$$

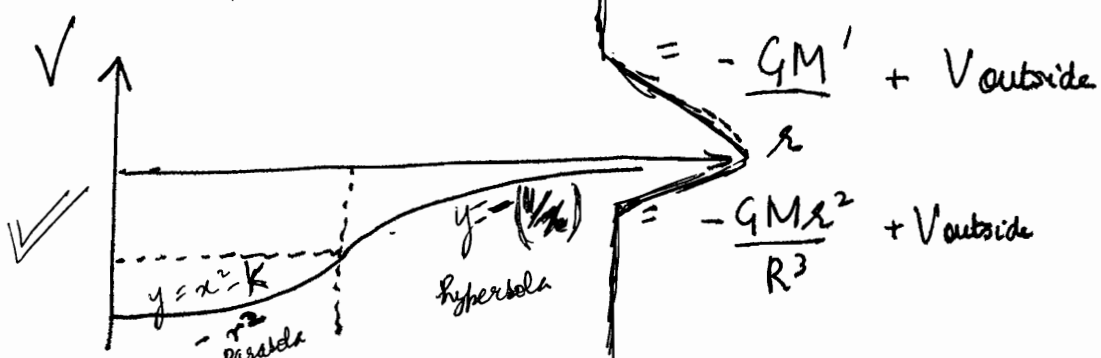
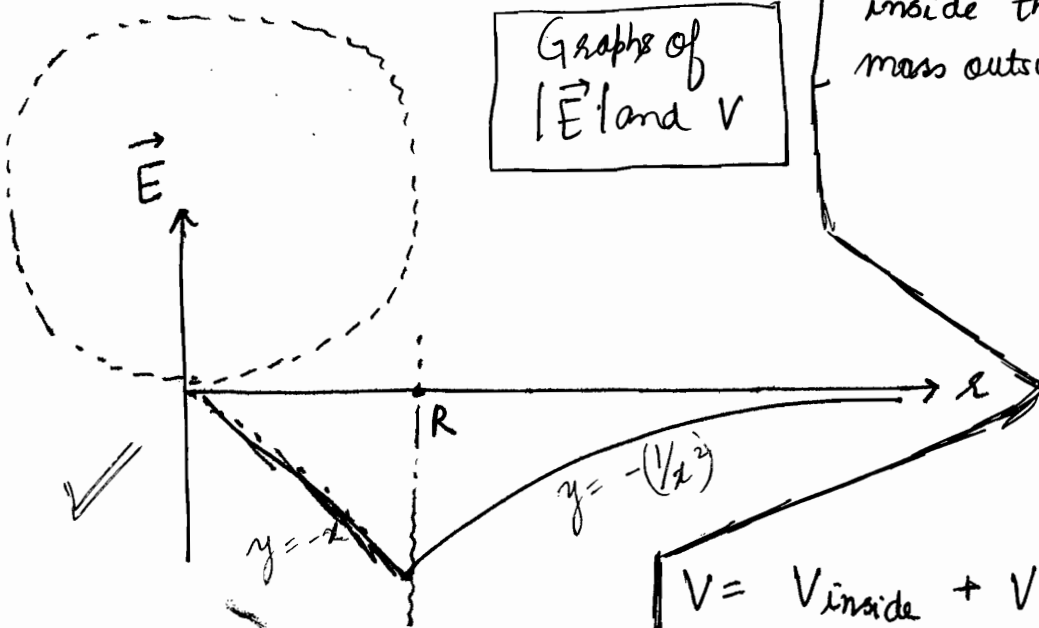


$$\Rightarrow \vec{E} = -\frac{GM}{R^3} \vec{r}$$

$$V = \int -\vec{E} \cdot d\vec{r}$$

Potential will due to mass inside the core as well as mass outside the core

Graphs of  $|\vec{E}|$  and  $V$



$$V = V_{\text{inside}} + V_{\text{outside}}$$

$$= -\frac{GM'}{r} + V_{\text{outside}}$$

$$= -\frac{GM r^2}{R^3} + V_{\text{outside}}$$

$$V_{\text{outside}} =$$

$$= - \int \vec{E} \cdot d\vec{e}$$

$$= + \int \frac{GM}{R^3} \vec{r} \cdot d\vec{e}$$

$$= \int_R^r \frac{GM}{R^3} \vec{r} \cdot d\vec{e}$$

$$= \frac{GM}{2R^3} [R^2 - r^2]$$

$$V_{\text{net}} = - \frac{GM}{2R^3} [2r^2 + r^3 - 2R^2]$$

$$= - \frac{GM}{2R^3} [3r^2 - R^2] \Rightarrow - \frac{GM}{R} - V_1 = + \int_r^R \frac{GM}{R^3} \vec{r} \cdot d\vec{e}$$

$$- \frac{GM}{R} - V_1 = \frac{GM}{2R^3} [R^2 - r^2]$$

$$\Rightarrow V_1 = - \frac{GM R^2}{R^3} - \frac{GM}{2R^3} [R^2 - r^2]$$

$$= - \frac{GM}{2R^3} [3R^2 - r^2]$$

PTO, PTO

We know field b/w internal points

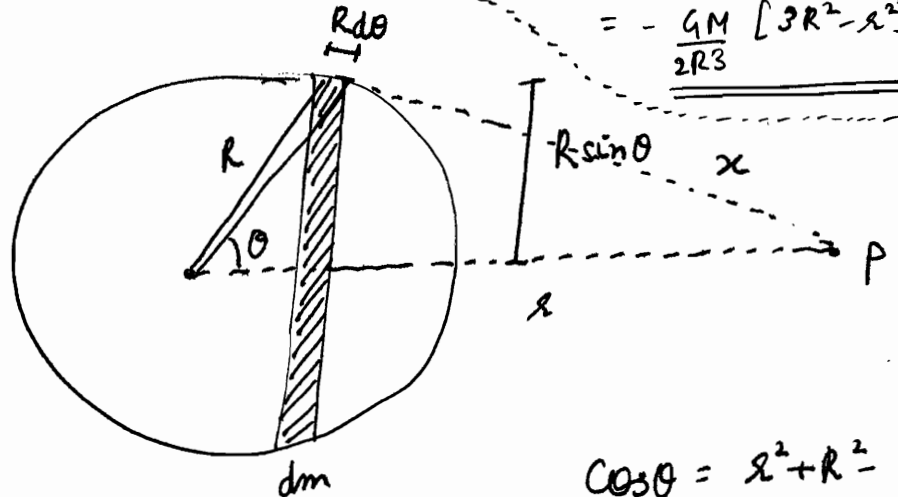
$$\Rightarrow \int_{V_1}^{V_2} dV = - \int_r^R E \cdot d\vec{e}$$

Gauss law for spherical shell

$$\sigma = \frac{M}{4\pi R^2}$$

$$dM = \sigma \cdot 2\pi R^2 \sin\theta d\theta$$

$$= \frac{M}{2} \sin\theta d\theta$$



$$\cos\theta = \frac{x^2 + R^2 - r^2}{2Rx}$$

$$- \sin\theta d\theta = \frac{x dx}{rR}$$

All masses at distance \$x\$

$$\rightarrow dV = - \frac{G dm}{x}$$

$$= - \frac{GM}{2} \frac{\sin\theta d\theta}{x} = + \frac{GM}{2} \frac{x dx}{rR x}$$

(i) If P is outside

$$dV = - \frac{GM}{2Rr} dx$$

← since this is not, only for outside  $R$ , it is valid for all  $x$ .

For outside points

$$\int dV = \int_{r-R}^{r+R} - \frac{GM}{2Rr} dx$$

$$V = - \left( \frac{GM}{R} \right)$$

$$\vec{E} = - \vec{\nabla} V = - \frac{dV}{dr} = - \frac{GM}{r^2} \hat{r}$$

$$\vec{E} = - \frac{GM}{r^2} \hat{r}$$

For inside points

$$dV = - \frac{GM}{2Rr} dx$$

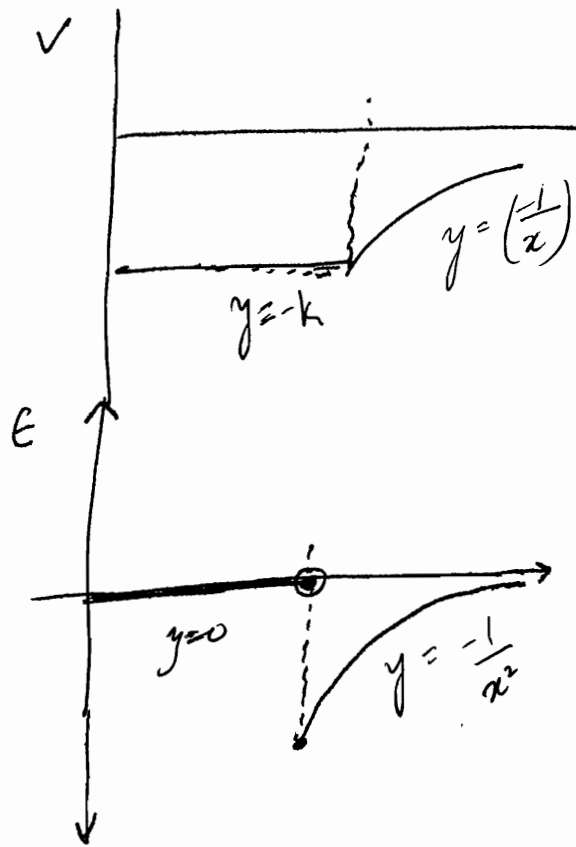
$$V = - \frac{GM}{2Rr} \int_{R-r}^{R+r} dx$$

$$V = - \left[ \frac{GM}{R} \right]$$

$\Rightarrow$  Equipotential inside the shell

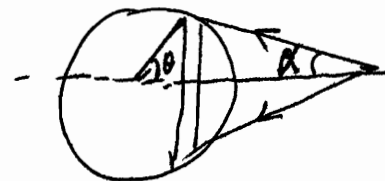
$$\Rightarrow \vec{E} = - \vec{\nabla} V = 0$$

$\Rightarrow$  NO field inside the shell



Graphs of  
V and E

If to be found out via  $\vec{E}$



$$E = \int dE \cos \alpha$$

$$= \int -\frac{GM}{2} \frac{\sin \theta d\theta}{x^2} \cos \alpha$$

$$\cos \alpha = \frac{x^2 + r^2 - R^2}{2xr}$$

$$= -\int \frac{GM}{2} \frac{x dx}{xR + x^2} \cos \alpha$$

$$= -\frac{GM}{4R^2R} \int_{R+R}^{r+R} \frac{x^2 + r^2 - R^2}{2x^2} dx = -\left(\frac{GM}{R^2}\right)$$

$$= -\frac{GM}{4R^2R} \left[ 2R + \int_{r-R}^{r+R} \frac{x^2 - R^2}{x^2} dx \right]$$

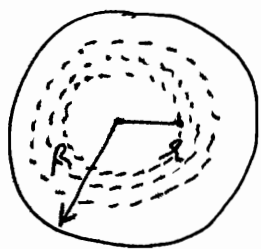
$$= -\frac{GM}{4R^2R} [2R + 2R] = -\left(\frac{GM}{R^2}\right)$$

⊙ If we now take limits as  $(R-x)$  to  $(R+x)$ , we get  $\vec{E} = 0$

Now  $V = \text{const.}$  found by boundary condition

⇒ Preferable to find 1<sup>st</sup> V, then  $E = -\nabla V$

①  $V_{\text{outside}}$  for a sphere wrt. a point lying inside sphere HW  
= ??



The sphere can be thought of to be made up of 1 inner core of radius  $r$  and ~~a~~ <sup>multiple</sup> layers of multiple concentric shells of radius  $x$  to  $R$ .

$$V_T = V_{\text{core}} + \sum V_{\text{shell}}$$

Let us take a shell at radius  $x$  from centre.

$$V_{\text{shell, inside}} = V_{\text{shell, surface}} = - \left[ \frac{GM'}{x} \right]$$

$$= - \frac{G \rho 4\pi x^2 dx}{x}$$

$$\Rightarrow V_T = V_{\text{core}} + \int_r^R -G 4\pi \rho x dx$$

$$= - \frac{GM r^2}{R^3} - \frac{3MG}{R^3} \frac{(R^2 - r^2)}{2}$$

$$= - \frac{GM}{2R^3} \left[ 2r^2 + 3R^2 - 3r^2 \right]$$

$$V_{\text{sphere}} = - \frac{GM}{2R^3} \left[ 3R^2 - r^2 \right]$$

## Solid Sphere

Outside from Gauss law

$$-E \cdot 4\pi R^2 = -4\pi G \cdot M$$

$$\Rightarrow E = \frac{GM}{R^2}, \quad \vec{E} = -\frac{GM}{R^2} \hat{r}$$

$$\vec{E} = -\nabla V = \nabla\left(\frac{GM}{R}\right)$$

$$\Rightarrow V = -\frac{GM}{R}$$

Surface

$$\vec{E} = -\frac{GM}{R^2} \hat{r}$$

$$V = -\frac{GM}{R}$$

Inside from Gauss law,

$$-E \cdot 4\pi R^2 = -4\pi G \frac{M}{R^3} \cdot R^3$$

$$\Rightarrow E = \frac{GM}{R^3} R, \quad \vec{E} = -\frac{GM}{R^3} R \hat{r}$$

$$E = -\nabla V$$

$$\Rightarrow dV = -E \cdot dr$$

$$\Rightarrow V_s - V_r = -\int_r^s E \cdot dr$$

$$\Rightarrow V_r = V_s + \int_r^s E \cdot dr$$

$$= -\frac{GM}{R} + \int_r^R \frac{GM}{R^2} R \cdot dr$$

$$= -\frac{GM}{R} - \frac{GM}{R^2} \cdot \frac{(R^2 - r^2)}{2}$$

$$= -\frac{GM}{2R^3} [2R^2 + R^2 - r^2]$$

$$V_r = -\frac{GM}{2R^3} [3R^2 - r^2]$$

## Shell

Outside

same

$$\vec{E} = -\frac{GM}{R^2} \hat{r}$$

$$V = -\frac{GM}{R}$$

Surface

same

$$\vec{E} = -\frac{GM}{R^2} \hat{r}$$

$$V = -\frac{GM}{R}$$

Inside

from Gauss law,

$$-E \cdot 4\pi r^2 = 0$$

$$\Rightarrow E = 0$$

$$dV = -\int_r^s E \cdot dr = 0$$

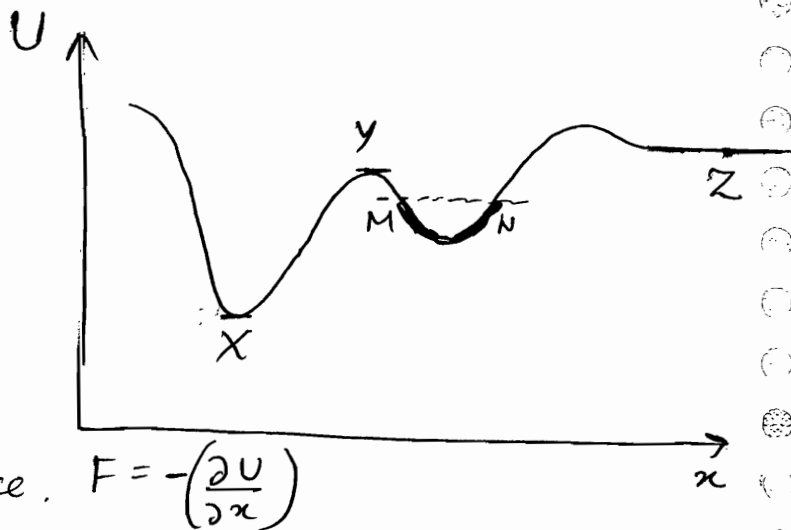
$$\Rightarrow V_s - V_r = 0$$

$$\Rightarrow V_r = V_s$$

$$\Rightarrow V_r = -\frac{GM}{R}$$

# Potential Energy Curve

⊙ A curve showing variation of Potential Energy of a particle with its position in the field is called Potential Energy Curve.



⊙ ~~Force~~  $F = -\nabla U$

For 1-d motion, slope of the curve gives the value of force.  $F = -\left(\frac{\partial U}{\partial x}\right)$

⊙ Force acting on the particle tends to pull it into a region of lower potential energy.

⊙ For points representing maxima or minima,  $F=0$  as slope = 0  $\Rightarrow$  Positions of Equilibrium

⊙ Y : Unstable Equilibrium. As slight movement left or right will cause a force that will further push the particle in the same direction

⊙ X : Stable Equilibrium. As slight movement left or right will give rise to a force that will cause the particle to return to X.

⊙ Z : neutral equilibrium. Constant Potential Energy region around it, hence no force acts on displacement.

⊙ Marked portion MN : Oscillations in the region. Such a region is called Bounded region or a Potential Well, and always exists about a point of minimum PE or stable equilibrium. The difference between top & bottom of well is called Binding Energy.



# MECHANICS (9)

24/11/11

## Rigid Body Dynamics

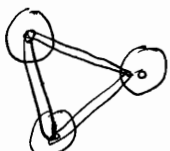
- RIGID BODY
- EULER'S ANGLES
- MOMENT OF INERTIA
- TORQUE-FREE MOTION SYM (Ω)
- TORQUE & PRESSION MOTION NON-SYM (Point)
- GYROSCOPE

### Rigid Body

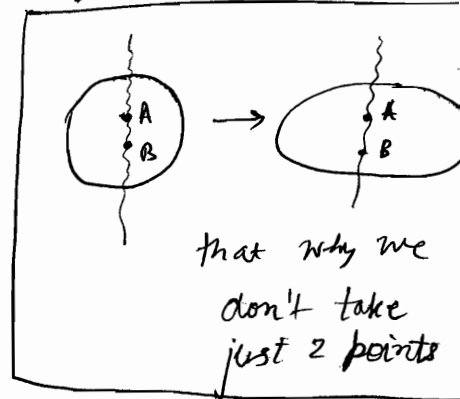
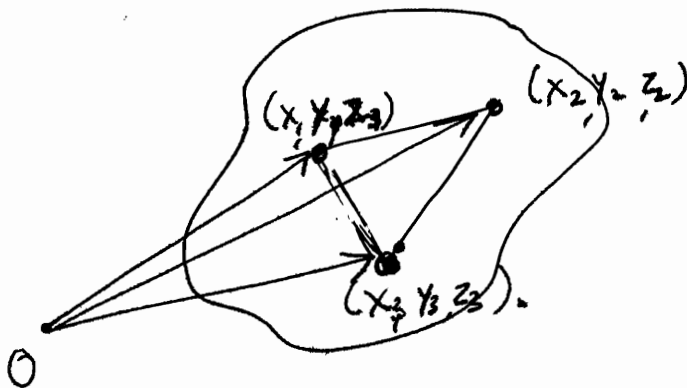
Interparticle distance should be constant.

Mathematically represented as 3 non-collinear points; keeping point mass at each point. If interparticle distance is const., then its a rigid body.

Remember Constraints



rigid body



No. of equations required for description of complete motion = No. of degrees of freedom.

Degrees of Freedom : Minimum number of independent coordinates required to describe the dynamical system.

Represented by  $f$ .

$f$  = Total coordinates required to describe the system - constants (constraints)

$$f = [3N - k]$$

↑ No. of Particles  
3d-motion

For a general rigid body,  $N=3$  by definition

$$f = 3 \times 3 - 3 = 6$$

out of 6,

3 coordinates are required for translation motion.

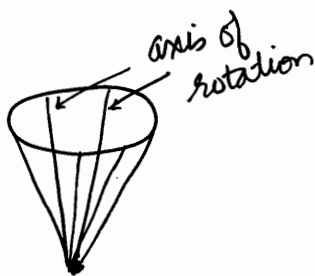
3 coordinates are required for rotational motion.

$$R_{cm} = \frac{\sum m_i r_i}{\sum m_i} \quad \frac{dP_{cm}}{dt} = F_{ext}$$

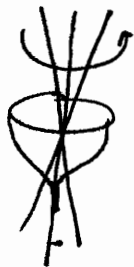


SPIN

[Axis passes through the body]



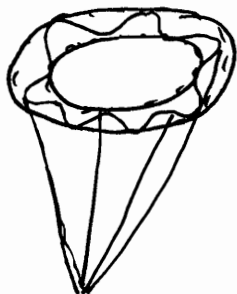
axis of rotation



PRECESSION

[rotation of axis of motion in a circle]

it happens when there is perpendicular component of Torque



NUTATION

[Oscillation of axis of rotation b/w 2 circles]

Generalized coordinates should be able to take care of all types of motion.

Refer WIKIPEDIA ARTICLE ON EULER ANGLES....

Euler's Angles : In Rotations performed in specified manner

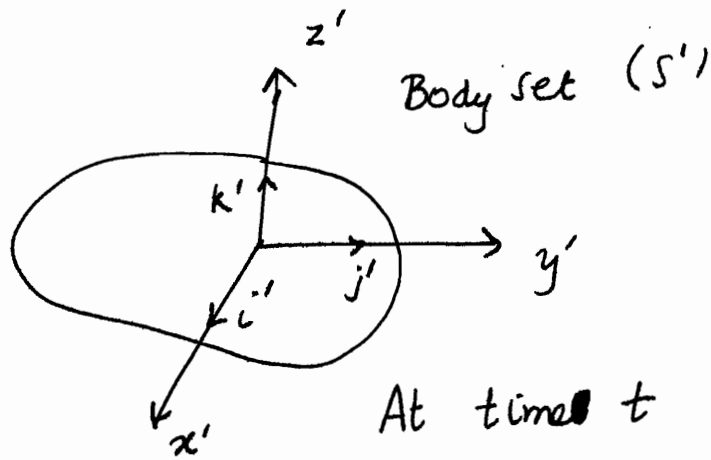
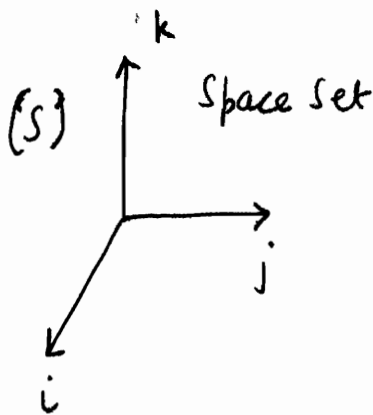
E.A. are used to describe the rotational motion of a rigid body.

Three generalized coordinates [i.e. three successive] rotation

$[\phi, \theta, \psi]$   
Phi Theta Psi

⊙ Euler Angles are 3 Angles that represent 3 rotations that move a reference frame to a given referred frame. It is equivalent to saying that any rotation matrix can be decomposed into a product of 3 elemental rotation matrices.

6 degrees of freedom :  $[x, y, z, \phi, \theta, \psi]$



At  $t=0$ , both  $S$  and  $S'$  coincide

$\vec{A}$  can be written in  $S$  or  $S'$

$$\vec{A} = (\vec{A} \cdot \hat{i}) \hat{i} + (\vec{A} \cdot \hat{j}) \hat{j} + (\vec{A} \cdot \hat{k}) \hat{k}$$

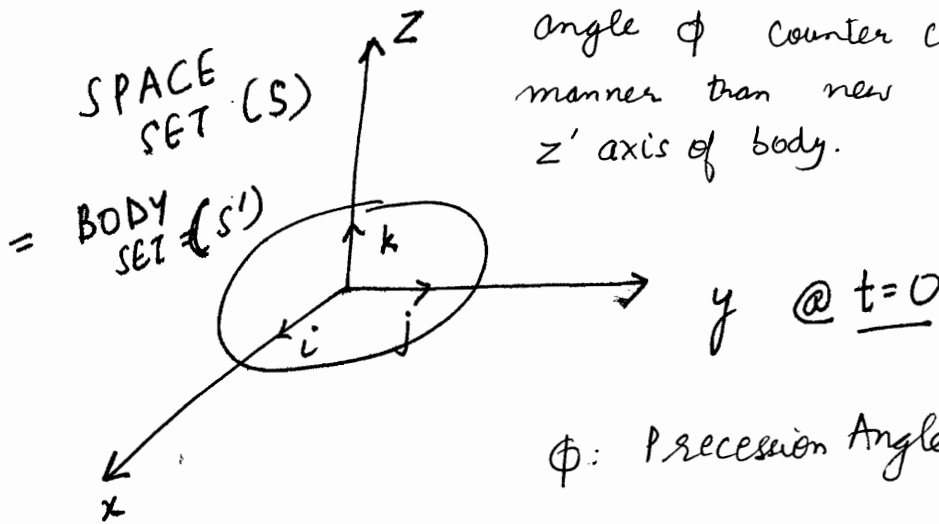
$$\vec{A} = (\vec{A} \cdot \hat{i}') \hat{i}' + (\vec{A} \cdot \hat{j}') \hat{j}' + (\vec{A} \cdot \hat{k}') \hat{k}'$$

Note that  $\hat{i}', \hat{j}', \hat{k}'$  are obtained from  $\hat{i}, \hat{j}, \hat{k}$  after performing all types of rotation.

Aim is to write

$$\begin{bmatrix} i' \\ j' \\ k' \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix}$$

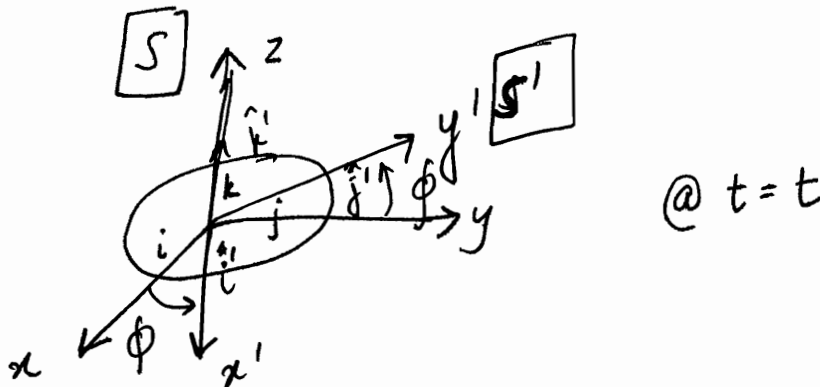
1<sup>st</sup> rotation



Space axes rotated about z-axis by angle  $\phi$  counter clockwise in such a manner that new YZ plane Y,Z, contains z' axis of body.

$\phi$ : Precession Angle

Let us perform the 1<sup>st</sup> rotation about z-axis of Space Coordinates by an angle  $\phi$ .



○ 3 choices: rotate about any axis

@ t = t

$$\hat{i}_1 = (\hat{i}_1 \cdot \hat{i}) \hat{i} + (\hat{i}_1 \cdot \hat{j}) \hat{j} + (\hat{i}_1 \cdot \hat{k}) \hat{k}$$

$$\hat{j}_1 = (\hat{j}_1 \cdot \hat{i}) \hat{i} + (\hat{j}_1 \cdot \hat{j}) \hat{j} + (\hat{j}_1 \cdot \hat{k}) \hat{k}$$

$$\hat{i}_1 = \cos \phi \hat{i} + \sin \phi \hat{j}$$

$$\hat{j}_1 = -\sin \phi \hat{i} + \cos \phi \hat{j}$$

$$\hat{k}_1 = \hat{k}$$

$$\Rightarrow \begin{bmatrix} \hat{i}_1 \\ \hat{j}_1 \\ \hat{k}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_D \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

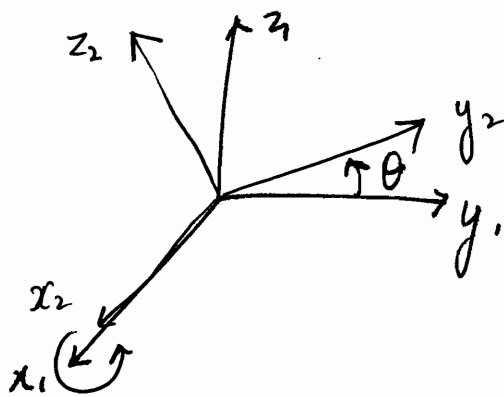
Note that  $[D]^{-T} = [D]$

## 2<sup>nd</sup> Rotation

2 choices: now x or y axis

Now we can perform rotation either x axis or y axis such that  $z_2$  will coincide with  $z'$  axis.

Hence rotate about  $x_1$  axis by angle  $\theta$ , so that,  $z_1$  axis after rotation, coincide with body's  $z'$  axis



$x, y, z$ , rotated about  $x'$  axis so that  $z_2 = z'$

$\theta$ : Nutation Angle

$$\hat{i}_2 = \hat{i}_1$$

$$\hat{j}_2 = \hat{j}_1 \cos \theta + \sin \theta \hat{k}_1$$

$$\hat{k}_2 = \cos \theta \hat{k}_1 - \sin \theta \hat{j}_1$$

$$\begin{bmatrix} \hat{i}_2 \\ \hat{j}_2 \\ \hat{k}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}}_C \begin{bmatrix} \hat{i}_1 \\ \hat{j}_1 \\ \hat{k}_1 \end{bmatrix}$$

Note that  $[C][C^T] = [I]$

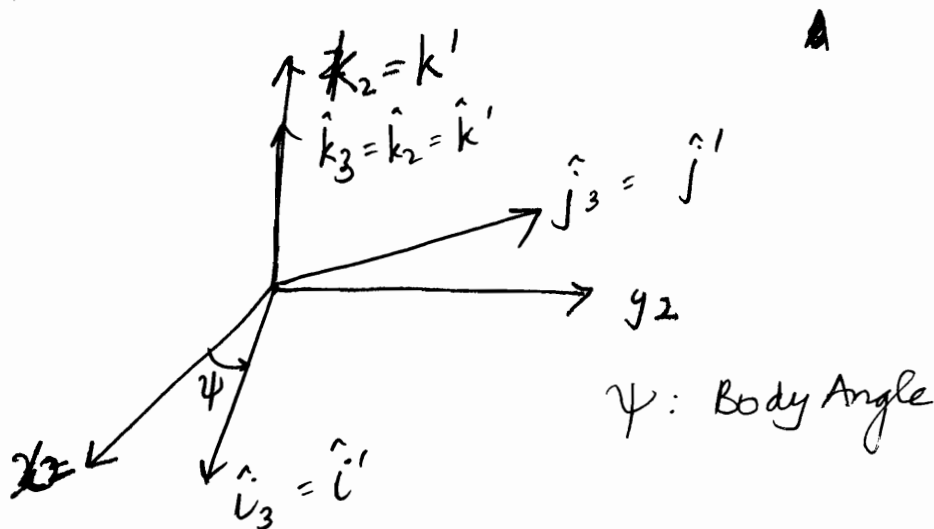
### 3<sup>rd</sup> rotation

Rotation about  $z_2$  or  $z'$  s.t.  $x_2, y_2$  will become  $x', y'$  after the rotation.

This motion is SPIN. [rotation about body's axis]

[No freedom or choice: only rotation about  $z'$ ]

Hence 3<sup>rd</sup> rotation, performed about  $z_2$  axis ( $z'$ ) by  $\psi$ , s.t.  $[x_3, y_3]$  ~~axis~~ axis coincides with  $[x', y']$  axis.



$$\begin{aligned} \hat{i}_3 &= \cos\psi \hat{i}_2 + \sin\psi \hat{j}_2 \\ \hat{j}_3 &= -\sin\psi \hat{i}_2 + \cos\psi \hat{j}_2 \\ \hat{k}_3 &= \hat{k}_2 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} \hat{l}_3 \\ \hat{j}_3 \\ \hat{k}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_B \begin{bmatrix} \hat{l}_2 \\ \hat{j}_2 \\ \hat{k}_2 \end{bmatrix}$$

Now

$$\begin{bmatrix} \hat{l}'_1 \\ \hat{j}'_1 \\ \hat{k}'_1 \end{bmatrix} = \begin{bmatrix} \hat{l}_3 \\ \hat{j}_3 \\ \hat{k}_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \hat{l}'_1 \\ \hat{j}'_1 \\ \hat{k}'_1 \end{bmatrix} = [B][C][D] \begin{bmatrix} \hat{l}_1 \\ \hat{j}_1 \\ \hat{k}_1 \end{bmatrix}$$

$$[A] = [B][C][D]$$

In general

$$[M][I] = [I][M] = 0$$

$$\Rightarrow \begin{bmatrix} \hat{l}'_1 \\ \hat{j}'_1 \\ \hat{k}'_1 \end{bmatrix} = [A] \begin{bmatrix} \hat{l}_1 \\ \hat{j}_1 \\ \hat{k}_1 \end{bmatrix}$$

Orthogonal Matrix:  $[A][A^T] = I$

For all B, C, D, A  $[M][M^T] = I$

$$A^T = [B][C][D]^T = [D^T][C^T][B^T]$$

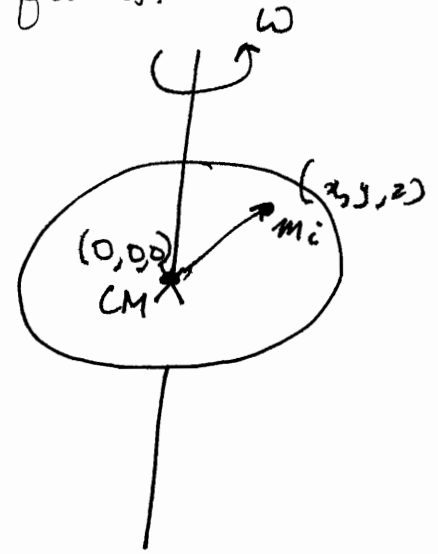
$$AA^T = [B][C][D][D^T][C^T][B^T] = I$$

- All these transformations are
- i) linear
  - ii) orthogonal
- ⊛ Finite rotations can't be represented as vectors since they don't follow vector addition rule. However, infinitesimal rotations are vectors. Hence  $\omega$  is vector.

## Moment of Inertia

Take any axis passing through COM.

⊛  $\vec{\omega}$  (instantaneous) is fixed in all frames.



$$I_{CM} = \sum m_i r_i^2 \quad \text{def}^n$$

In space set,

$$L = \left( \frac{d\vec{J}}{dt} \right)$$

In Body set,  $N = \left( \frac{d\vec{J}}{dt} \right)_R + [\vec{\omega} \times \vec{J}]$

$$\vec{J} = \sum \vec{r}_i \times \vec{p}_i$$

$$\vec{\tau} = \frac{d\vec{J}}{dt} = \sum \vec{r}_i \times \vec{F}$$

$$\vec{J} = \sum \vec{r}_i \times m_i \vec{v}_i = \sum \vec{r}_i \times m_i (\vec{\omega} \times \vec{r}_i)$$

$$= \sum m_i (\vec{r}_i \times (\vec{\omega} \times \vec{r}_i))$$

$$= \sum m_i [\vec{\omega} |\vec{r}_i|^2 - \vec{r}_i [\vec{\omega} \cdot \vec{r}_i]]$$



$$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

$$\Rightarrow \vec{J} = \sum m_i \left[ (x_i^2 + y_i^2 + z_i^2) [\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}] - (x_i \hat{i} + y_i \hat{j} + z_i \hat{k}) (\omega_x x_i + \omega_y y_i + \omega_z z_i) \right]$$

~~scribbled out text~~

$$\Rightarrow \vec{J} = \sum m_i \left[ \hat{i} \left\{ (x_i^2 + y_i^2 + z_i^2) \omega_x - (\omega_x x_i + \omega_y y_i + \omega_z z_i) x_i \right\} + \hat{j} \left\{ (x_i^2 + y_i^2 + z_i^2) \omega_y - (\omega_x x_i + \omega_y y_i + \omega_z z_i) y_i \right\} + \hat{k} \left\{ (x_i^2 + y_i^2 + z_i^2) \omega_z - (\omega_x x_i + \omega_y y_i + \omega_z z_i) z_i \right\} \right]$$

no need to write this huge expression.

$$\Rightarrow \underline{J_x} = \sum m_i \left\{ (y_i^2 + z_i^2) \omega_x - x_i y_i \omega_y - x_i z_i \omega_z \right\}$$

write  $J_x$

$$\underline{J_x} = \sum m_i (y_i^2 + z_i^2) \omega_x - \sum m_i x_i y_i \omega_y - \sum m_i x_i z_i \omega_z$$

$$J_x = I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z$$

$$J_y = I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z$$

$$J_z = I_{zx} \omega_x + I_{zy} \omega_y + I_{zz} \omega_z$$

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$\vec{J} =$  [Moment of Inertia Tensor]  $\vec{\omega}$

moments of inertia

$$\begin{cases} I_{xx} = \sum m_i (y_i^2 + z_i^2) \\ I_{yy} = \dots \\ I_{zz} = \dots \end{cases}$$

Products of inertia

$$\begin{cases} I_{xy} = I_{yx} = -\sum m_i x_i y_i \\ I_{yz} = I_{zy} = \dots \\ I_{zx} = I_{xz} = \dots \end{cases}$$

Cross product of inertia

Maximum no. of linearly independent rows!!

It is a symmetric tensor of rank 2.

Tensor is a transformation of coordinates

$$\vec{J} = \overset{\leftarrow}{I} \vec{\omega}$$

$$\overset{\leftarrow}{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

If continuous mass distribution

$$I_{xx} = \int dm (y_i^2 + z_i^2)$$

$$\text{Kinetic Energy} = \frac{1}{2} \vec{\omega} \cdot \vec{J}$$

$$= \frac{1}{2} \vec{\omega} \cdot \overset{\leftarrow}{I} \cdot \vec{\omega}$$

⊙ Note that 'I' is not a number but a transformation matrix.

In normal experiences,  $\omega_y$  and  $\omega_x$  are 0 and body is symmetric i.e.  $I_{xz}, I_{yz}$  are 0

Hence,  $\vec{J} = I_{zz} \vec{\omega}$  hence we think  $I = I_{zz}$  (in this case) is a number

If body rotating about z axis, i.e.  $\omega_x = \omega_y = 0$

$$\Rightarrow J_x = I_{xz} \omega_z$$

$$J_y = I_{yz} \omega_z$$

$$J_z = I_{zz} \omega_z$$

⊙ Note that in translatory motion  $\vec{v} \parallel \vec{p}$

$$\Rightarrow \frac{1}{2} \vec{v} \cdot \vec{p} = \frac{1}{2} m v^2$$

But in rotation motion,  $\vec{J} \not\parallel \vec{\omega}$  (in general)

$$\Rightarrow K.E. = \frac{1}{2} \vec{\omega} \cdot \vec{J}$$

If symmetric body  $\begin{cases} \rightarrow \text{Cross products vanish about principal axis} \\ \rightarrow I_{xx} = I_{yy} \neq I_{zz} \end{cases}$

$$\Rightarrow \sum x_i y_i = 0$$

$$\sum x_i z_i = 0$$

$$\sum z_i y_i = 0$$

$\Rightarrow$  Cross Products vanish.

We need to find

Principal Axes of Symmetry

$\leftarrow$  def<sup>n</sup>

About these axis, Cross products vanish

$\swarrow$  we refer to them as

$\Rightarrow$  we get Principle Moment of Inertia.  $x, y, z$

i.e.

$$\vec{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

In this case,  $J_x = I_{xx} \omega_x$

$$J_y = I_{yy} \omega_y$$

$$J_z = I_{zz} \omega_z$$

If rotation along 1 Principle Axis, for a symmetric body  
only  $J_z = I_{zz} \omega_z$

In order to reduce calculation, we need to locate  
Principle Axis of Rotation.

How to locate??

Either locate by inspection [in most of cases]

If unable to locate by inspection, we can find by following

Along  
Principle  
Axis

$$I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z = I \omega_x$$

$$\Rightarrow (I_{xx} - I) \omega_x + I_{xy} \omega_y + I_{xz} \omega_z = 0 \quad \text{--- (1)}$$

Similarly

$$I_{yx} \omega_x + (I_{yy} - I) \omega_y + I_{yz} \omega_z = 0 \quad \text{--- (2)}$$

$$I_{zx} \omega_x + I_{zy} \omega_y + (I_{zz} - I) \omega_z = 0 \quad \text{--- (3)}$$

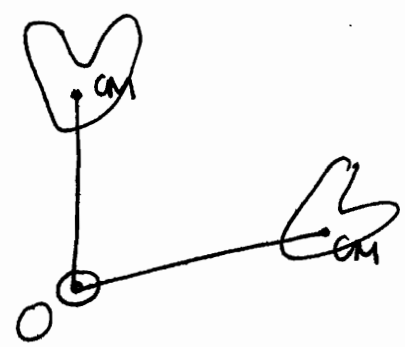
★ In order to have solutions, this determinant has to be 0 as all other determinants are zero.

$$\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = 0$$

Secular Equation of Inertia

Solution will give 3 values of  $I$ , namely  $I_1, I_2, I_3$   
These are Principal Moment of Inertia.

⊙ Rotations around the x, y, and z axis are called Principal Rotations. Rotation around any axis can be performed by taking a rotation about x-axis, followed by a rotation about y-axis and followed by a rotation around z-axis. In short, any spatial rotation can be decomposed into a combination of principal rotations.



Rotation about O can be considered as translation of CM about O and rotation of body around C.M.  
(जय B.M. Sharma)

⊙ If along some axis z,  $I_{xz}$ ,  $I_{yz}$  and  $I_{zz}$  exist and  $\vec{\omega}$  is along z,  $\Rightarrow$

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \begin{bmatrix} - & - & I_{xz} \\ - & - & I_{yz} \\ - & - & I_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega_z \end{bmatrix}$$

$$\vec{J} = I_{xz} \omega_z \hat{i} + I_{yz} \omega_z \hat{j} + I_{zz} \omega_z \hat{k}$$

Hence  $\vec{J}$  is NOT ALWAYS along  $\vec{\omega}$ .

★ The total angular momentum of the system about the fixed or the reference points is the vector sum of the angular momentum of COM about that point and angular momentum of system about C.O.M.  
 Former is called orbital angular momentum and the latter is called spin angular momentum.

★ For moon,  $J_{\text{spin}} = \frac{2}{5} M R_{\text{moon}}^2 \omega_{\text{rot}}$ .  $J_{\text{orbit}} = M R_{\text{earth-moon}}^2 \omega_{\text{rev}}$   
 $\omega_{\text{spin}} = \omega_{\text{revolution}}$  for moon : that's why we see the same face of moon always. In order to see both faces of the moon,  $\omega_{\text{spin}} = \frac{1}{2} \omega_{\text{rev}}$  or  $\omega_{\text{rev}} = \frac{1}{2} \omega_{\text{spin}}$

○ In an inertial frame of reference,

$$\left(\frac{dL}{dt}\right)_s = \left(\frac{d}{dt} I_s \omega\right)_s = (\tau)_s$$

where  $I_s$  is moment of inertia calculated in the inertial frame. Although this law is universally true, it is not always helpful in solving for the motion of a general rotating rigid body since  $I_s$  will change during the motion.

Therefore we change to coordinate frame fixed in the rotating ~~frame~~ body, and chosen so that its axes are aligned with principle axes of Moment of Inertia Tensor. Since  $I_R$  is fixed, it simplifies the calculations.

This change of frame gives rise to EULER'S EQUATIONS.

# MECHANICS (10)

25/11/11

From now on, the discussion will

In Principal axis, presume that  $\omega$ 's are along 3 principal

$$\vec{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

$$\vec{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

Rather, this can be the assumption always!!

$$\vec{J} = \begin{bmatrix} I_1 \omega_1 \\ I_2 \omega_2 \\ I_3 \omega_3 \end{bmatrix} \quad \left. \vphantom{\vec{J}} \right\} \text{ in Body set of Coordinates}$$

$$\left( \frac{d\vec{J}}{dt} \right)_S = \vec{N} = \left( \frac{d\vec{J}}{dt} \right)_R + \vec{\omega} \times \vec{J}$$

Usually  $\vec{L}$  equation  
1 fixed point on the body,  
say O, पर मजदारी है i.e.

$$\vec{N}_{ext, O} = \left( \frac{d\vec{L}_O}{dt} \right)_O = \left( \frac{d\vec{L}_O}{dt} \right)' + \vec{\omega} \times \vec{L}_O$$

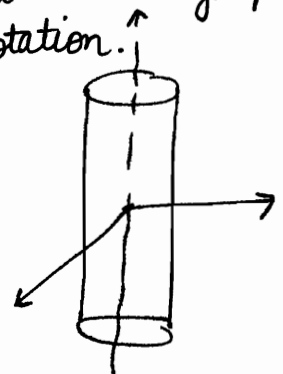
$$\vec{\omega} \times \vec{J} =$$

$$\begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ I_1 \omega_1 & I_2 \omega_2 & I_3 \omega_3 \end{vmatrix}$$

We assume 3 Principal Axis

$$\begin{aligned} N_1 &= I_1 \dot{\omega}_1 - \omega_2 \omega_3 [I_2 - I_3] \\ N_2 &= I_2 \dot{\omega}_2 - \omega_3 \omega_1 [I_3 - I_1] \\ N_3 &= I_3 \dot{\omega}_3 - \omega_1 \omega_2 [I_1 - I_2] \end{aligned}$$

These 3 equations are referred to as EULER'S EQUATIONS. They describe the rotation of a rigid body in a frame of reference fixed in the rotating body & having its axes parallel to the body's principle axes of rotation.



For Symmetric Body:

•  $I_{zz}$

•  $I_{xx} = I_{yy} \neq I_{zz}$

•  $I_{xz} = 0 = I_{yz}$   
 $= I_{xy} = 0$

Spherical Top:  $I_1 = I_2 = I_3$

Asymmetric body:  $I_1 \neq I_2 \neq I_3$

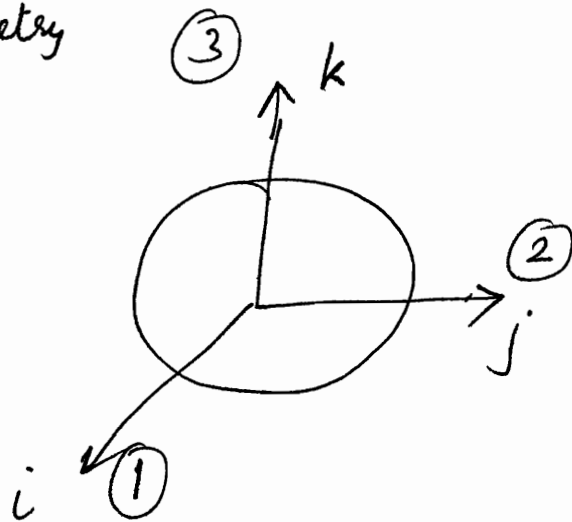
## Torque-free Motion of a symmetric body

No Torque is acting  $\Rightarrow N=0$

If no  $T$   
ACTING

let  $I_3$  be axis of symmetry

$$\Rightarrow I_1 = I_2$$



$$\Rightarrow I_1 \dot{\omega}_1 = (I_2 - I_3) \omega_2 \omega_3$$

$$I_2 \dot{\omega}_2 = (I_3 - I_1) \omega_3 \omega_1$$

$$I_3 \dot{\omega}_3 = (I_1 - I_2) \omega_1 \omega_2$$

[in absence of external intervention,  
energy remains conserved!!]

In such a motion, Rotational kinetic Energy is const.

Multiply (i) by  $\omega_i$  and adding

$$I_1 \omega_1 \dot{\omega}_1 + I_2 \omega_2 \dot{\omega}_2 + I_3 \omega_3 \dot{\omega}_3 = 0$$

$$\Rightarrow \frac{d}{dt} \left[ \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2 \right] = 0$$

expression for k.E.  $\Rightarrow \sum \frac{1}{2} I_i \omega_i^2 = \text{const.}$

$\Leftarrow$  We can get this from  
 $k.E. = \frac{1}{2} \vec{J} \cdot \vec{\omega}$



From this eqn, we get

$$\frac{\omega_1^2}{\left(\frac{2k}{I_1}\right)} + \frac{\omega_2^2}{\left(\frac{2k}{I_2}\right)} + \frac{\omega_3^2}{\left(\frac{2k}{I_3}\right)} = 1$$

: Moment of Inertia Ellipsoid

[POINSON'S SOLUTION]  
POINSON'S ELLIPSOID

If no  $\tau$  acting,  
& For a symmetric body,  $I_1 = I_2$

If no  $\tau$  acting  
on a SYMMETRIC  
BODY

From 3

$$I_3 \dot{\omega}_3 = 0$$

$$\Rightarrow \boxed{\omega_3 = \text{const}}$$

From 1

$$\dot{\omega}_1 = \left[ \frac{I_2 - I_3}{I_1} \omega_3 \right] \omega_2 = \Omega \omega_2$$

↑ const

$$\begin{aligned} \Omega &= \left[ \frac{I_2 - I_3}{I_1} \right] \omega_3 \\ &= \left[ 1 - \left( \frac{I_3}{I_1} \right) \right] \omega_3 \end{aligned}$$

$$\ddot{\omega}_1 = \Omega \dot{\omega}_2$$

From 2

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2} \omega_3 \omega_1$$

$$\dot{\omega}_2 = - \left[ \frac{I_1 - I_3}{I_2} \right] \omega_3 \dot{\omega}_1 = -\Omega \dot{\omega}_1$$

$$\ddot{\omega}_2 = -\Omega \dot{\omega}_1$$

$$\ddot{\omega}_1 = -\Omega^2 \omega_1$$

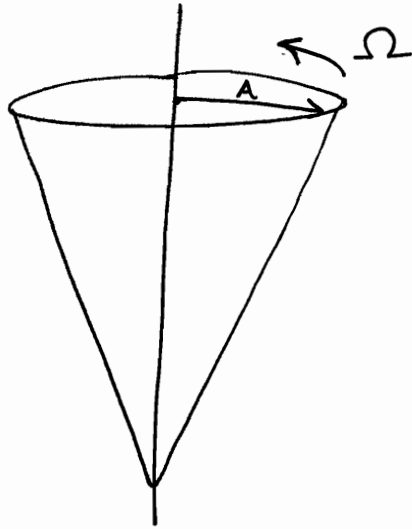
$$\Rightarrow \omega_1 = A \sin \Omega t$$

$$\ddot{\omega}_2 = -\Omega^2 \omega_2$$

$$\Rightarrow \omega_2 = A \cos \Omega t$$

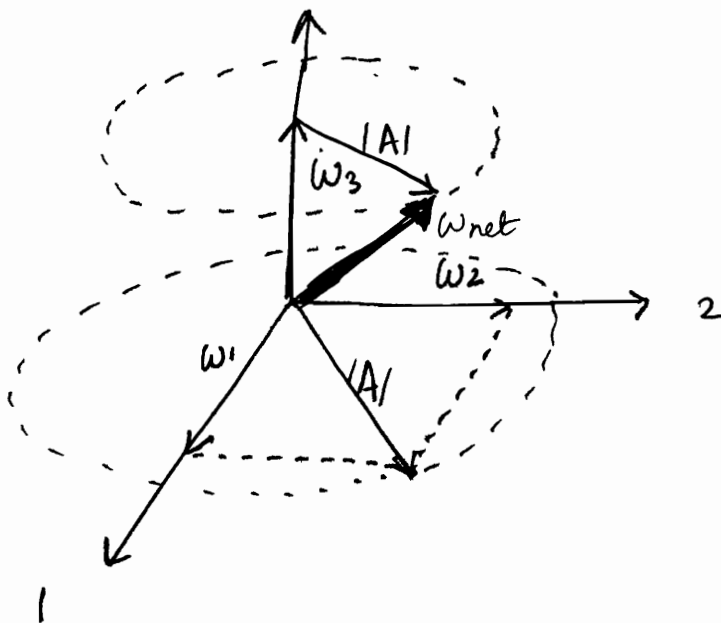
$$\omega_3 = \text{const.}$$

$$\vec{\omega} = A \sin \Omega t \hat{i} + A \cos \Omega t \hat{j} + \omega_3 \hat{k}$$



Precessional Angular Velocity =  $\Omega$

[It is a constant of motion of Torque free motion]



$$\Omega = \left[ \frac{I_1 - I_3}{I_1} \right] \omega_3$$

⊙ Earth also has precessional motion

$$\text{Time period precessional} = \frac{2\pi}{\Omega}$$

$$= \frac{2\pi}{(I_1 - I_3) \omega_3} I_1$$

$$= \frac{2\pi}{(I_1 - I_3)} T_{\text{day}} \cdot I_1$$

$$\left[ \omega_3 = \frac{2\pi}{T_{\text{day}}} \right]$$

$$= \left[ \frac{I_1}{I_1 - I_3} \right] T_{\text{day}}$$

$$\approx 300 T_{\text{day}} \approx \underline{\underline{300 \text{ days}}}$$

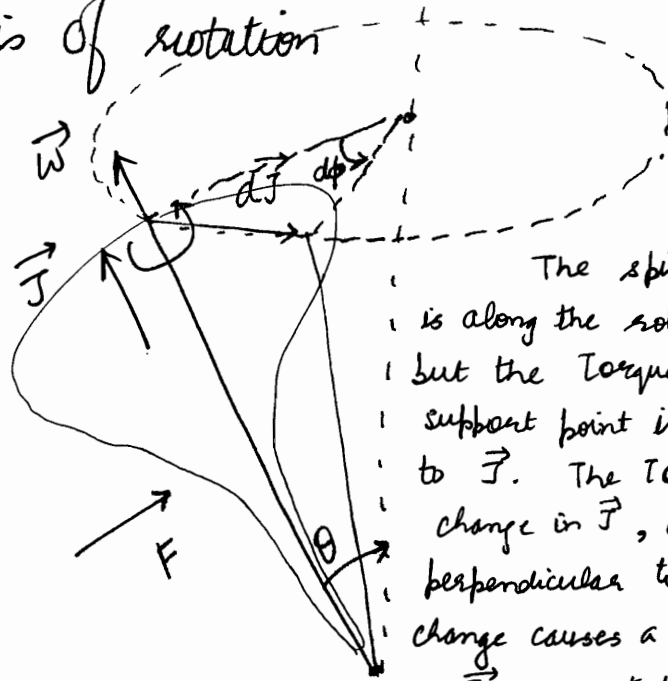
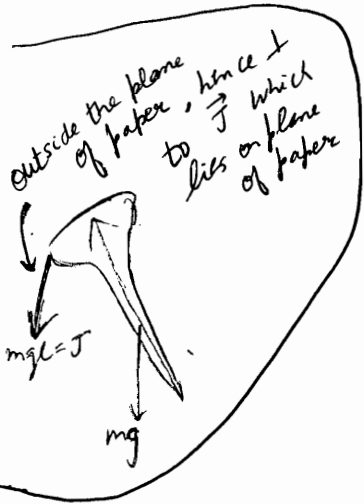
Precise value = 330 days

- ⊙
- 3 distances fixed : 6 degrees of freedom
  - 1 Point fixed : 3 degrees of freedom [No Translation Motion]
  - 2 Points fixed : 1 degree of freedom

⊙ So far we have calculated the precessional velocity when no torque is acting (Torque-free motion)  
Now we will calculate the precessional velocity caused due to external torque.

# Precession of a symmetrical top [with Torque]

Rotation of axis of rotation



The spin Angular Momentum ( $\vec{J}$ ) is along the rotation axis but the Torque about the support point is in direction  $\perp$  to  $\vec{J}$ . The Torque produces a change in  $\vec{J}$ ,  $d\vec{J}$ , which is perpendicular to  $d\vec{J}$ . Such a change causes a change in direction of  $\vec{J}$  but not in  $|\vec{J}|$ . This circular motion is called Precession.

For a symmetric body,

Symmetric  $\Rightarrow I_1 = I_2 = I_3 = I$   $\Rightarrow \vec{J} = I\vec{\omega}$  since symmetric about  $\hat{z} \Rightarrow I_x = I_y = 0$   $\Rightarrow \vec{J} = \omega_z I_{zz} \hat{z}$  [I about  $\omega$  axis]

Precessional Torque  $= \vec{\tau} = \left( \frac{d\vec{J}}{dt} \right)$

$\vec{\tau} \perp \vec{\omega}$   
 i.e.  $\vec{\tau} \perp \vec{J}$  [ $\vec{\omega} \parallel \vec{J}$ ]  
 i.e.  $d\vec{J} \perp \vec{J}$  [ $\vec{\tau} \parallel d\vec{J}$ ]

$$\left( \frac{d\phi}{dt} \right) = \vec{\Omega}$$

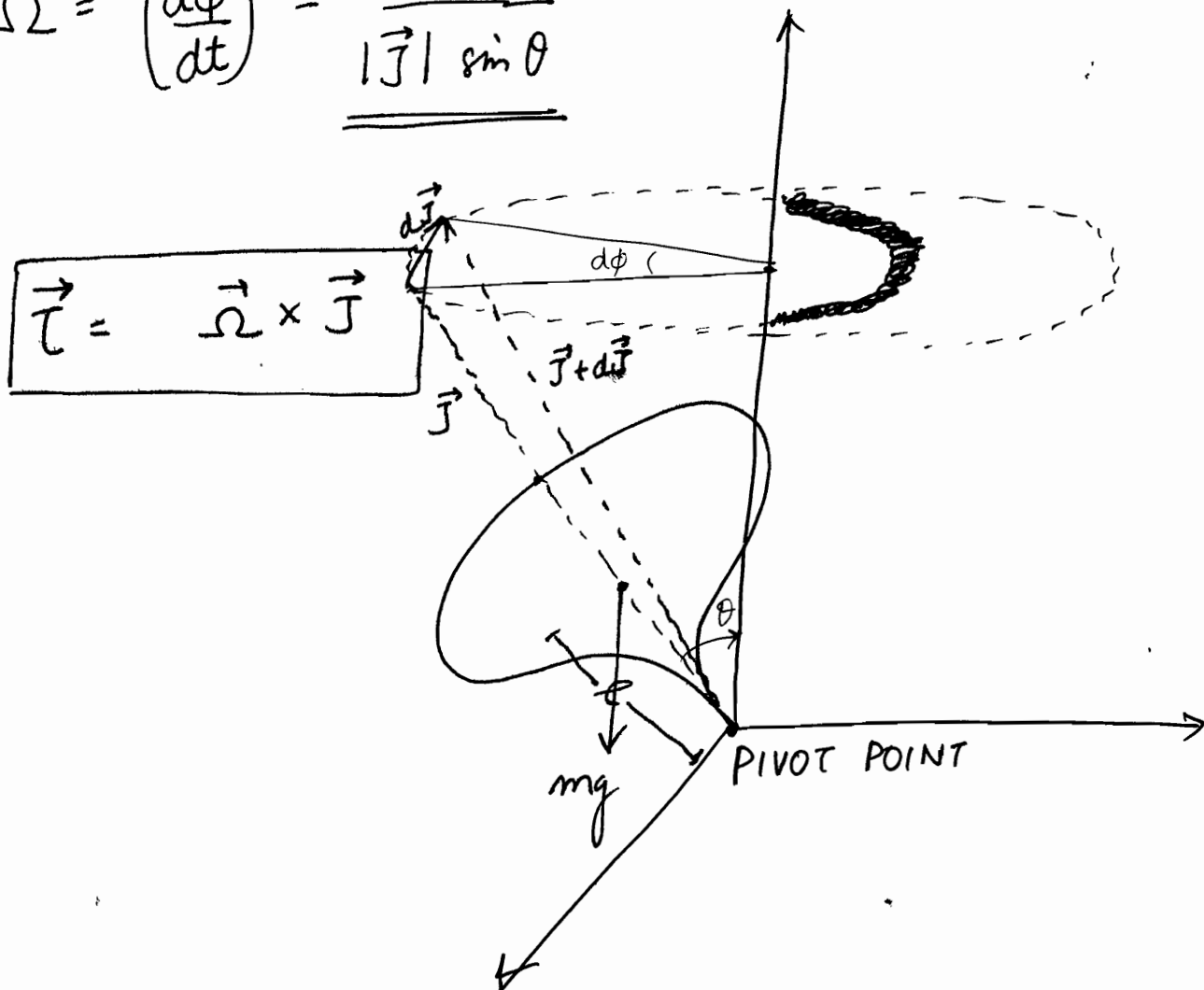
$\Rightarrow J$  is tracing out a circle.

$$\Delta\phi = \frac{\Delta\vec{J}}{|\vec{J}|\sin\theta}$$

$$\frac{\Delta\phi}{\Delta t} = \frac{\Delta\vec{J}/\Delta t}{|\vec{J}|\sin\theta} = \frac{\vec{\tau}}{|\vec{J}|\sin\theta}$$

Taking limit

$$\Omega = \frac{d\phi}{dt} = \frac{\tau}{\underline{\underline{|\vec{J}| \sin \theta}}}$$



$$d\phi = \frac{dJ}{J \sin \theta}$$

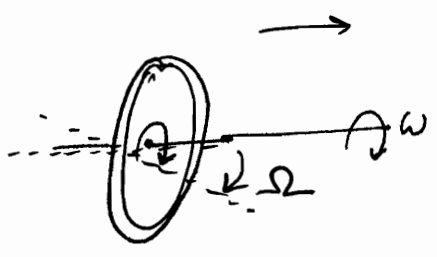
$$\Omega = \frac{(dJ/dt)}{J \sin \theta} = \frac{\tau}{J \sin \theta} = \frac{mgl \sin \theta}{J \sin \theta}$$

$$= \left[ \frac{mgl}{J} \right] = \left[ \frac{mgl}{I_{cm} \omega} \right]$$

$$\Rightarrow \boxed{\Omega = \frac{mgl}{I_{cm} \omega}}$$

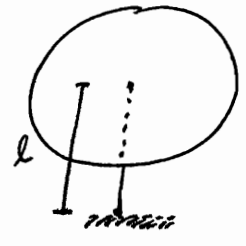
$\Rightarrow \Omega \propto [1/\omega]$  ✓  
 $\Rightarrow \Omega$  independent of  $\theta$  ✓

For ~~manoeuvring~~ <sup>manoeuvring</sup> motion of transport vehicles,  
Precession Torque is required



**Tut 4**

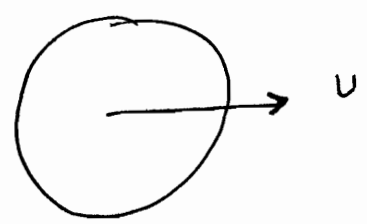
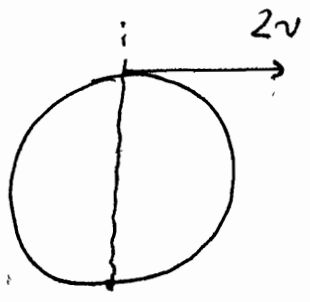
Q1)



⊛ Note that when frame is rotated, due to correspondingly opposite rotation of gimbals, the orientation of  $\vec{v}$  of flywheel remains same.

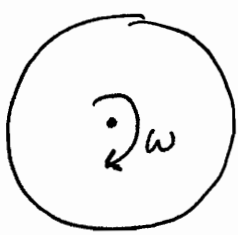
The change in orientation of spin axis is much less due to large  $\omega$  and large  $I$  of the rotating rotor

Q4



Translation of C.M.

+

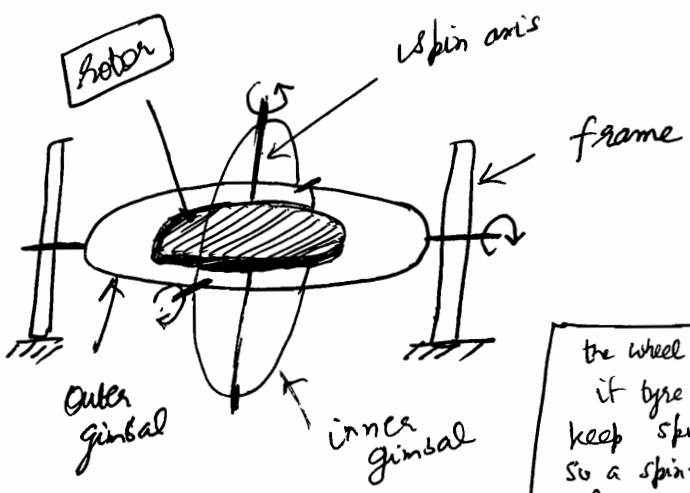


Rotation about C.M.

$v = \omega r$

This phenomenon is also used in bullet

→ →  
for precision of motion.



**Gyroscope**

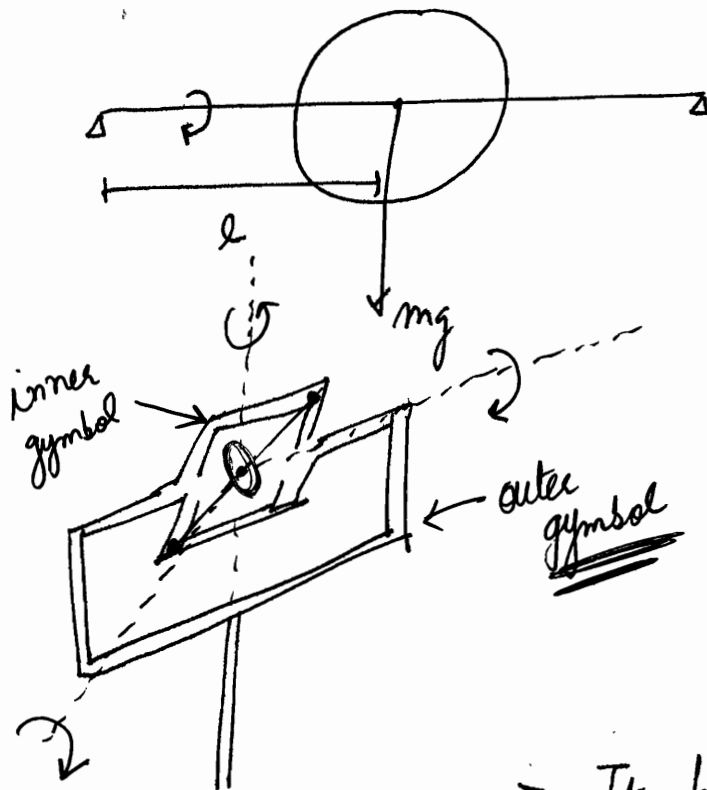
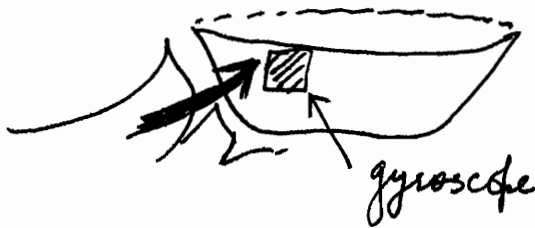
Pick up front tyre of a bicycle & spin the tyre very fast. Once the tyre is spinning you will find that turning

the wheel is more difficult using the handlebar than if tyre was not spinning. Spinning wheel wants to keep spinning along same axis. This is gyroscopic effect. So a spinning bullet does not want to tilt away from its axis of spin because of that.

# Gyroscope or Gyrostat

Gyroscope is a very heavy flywheel which can be rotated about any 3 mutually perpendicular directions i.e. it can be precessed.

It acts as direction stabilizer in boats, ships and aeroplanes.



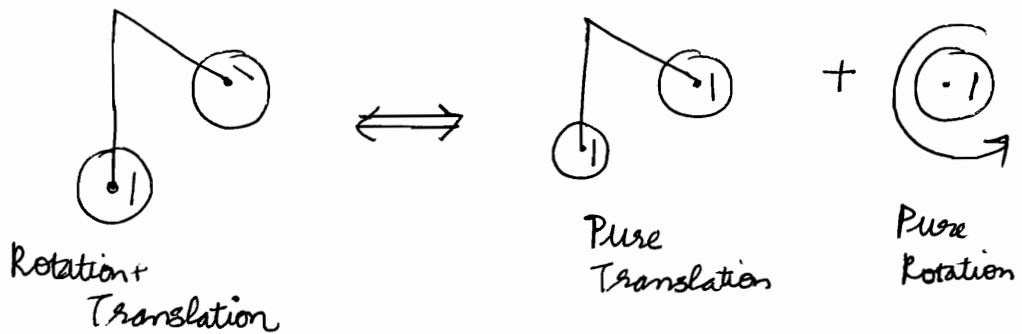
$$\Omega = \left( \frac{mgl}{I\omega} \right)$$

→ It produces a countering Torque to that applied by waves / currents and thereby stabilizing the motion.

Radius of Gyration (K)  $I = MK^2$

The distance from the axis of rotation, at which if its entire mass (M) is supposed to be concentrated, its moment of Inertia about the given axis would be same as with its actual distribution of mass.

- (\*) Rotation of body about the fixed axis through P with angular velocity  $\omega$  = its pure rotation about C.M. with same angular velocity ' $\omega$ ' + Its pure translation of its COM with linear velocity  $v = R\omega$



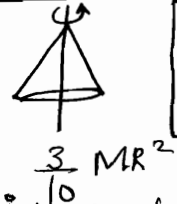
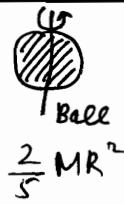
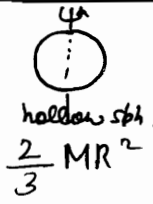
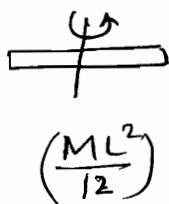
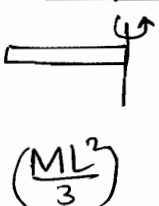
- (\*) A solid and hollow sphere of exactly same size & mass can be distinguished by rolling them on an inclined surface.

$$a = \left[ \frac{g \sin \theta}{1 + \frac{I}{MR^2}} \right]$$



$I_{shell} > I_{sphere}$

- (\*) The change in direction of rotation-axis, under the action of a constant torque perpendicular to the axis of rotation is called precession.



Moment of Inertia



Q) A bicycle wheel of mass 2 kg and radius 50 cm is rolling along on a road at 20 km/hr. What torque will have to be applied to handle to turn it through half a radian in .1 sec.

A)  $\omega_p = \left( \frac{T}{I\omega} \right)$ , Now  $\omega_p = \frac{0.5}{0.1} = 5 \text{ rad/sec.}$

$T = I\omega \cdot 5 \Rightarrow I = mR^2 \quad \omega = \left( \frac{v}{R} \right)$

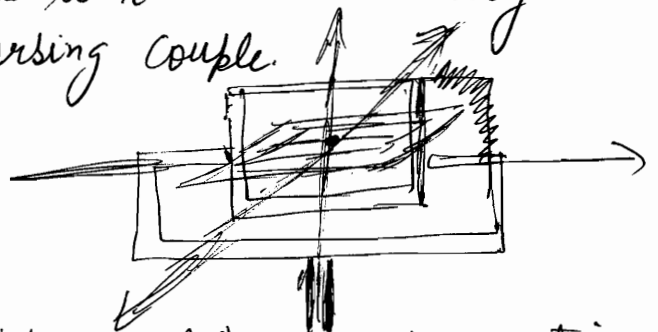
$\Rightarrow T = 2.78 \times 10^8 \text{ gm} \cdot \text{cm}^2/\text{sec}$

## Gyroscope

A gyroscope is a heavy circular disc (or a flywheel) whose axes of rotation can be turned about any of the three mutually perpendicular ~~directions~~ directions. The wheel is set in motion at very high speed. Since the rate of precession is inversely proportional to moment of inertia and the angular velocity about the axis of the wheel, the massive disc rotating at high speed makes for a great stability of the axes ~~of~~ of rotation, because of very small precession.

A gyroscope is used for steadying of motion and ensuring direction stability, the underlying principle being that the gyroscope precesses in such a manner that the counter couple due to the disc exactly ~~balances~~ balances the disturbing couple.

< draw the diagram >



### Applications

- ① Gyro Compass in planes & ships : Wheel set in motion

With its axis in direction of the magnetic meridian. The high speed of rotation ensures direction stability. The axis lies in the same direction irrespective of orientation of plane/ship. Hence useful as a compass.

② Stability of ships : Whenever the ship is about to topple due to torque of sea waves, the precession of the gyroscope gives rise to a counter torque that balances the destabilizing torque, thus stabilizing the ship.

Q/ Prove  $T = \frac{I\omega^2}{2}$  where  $I$  is moment of inertia about  $\vec{\omega}$  axis.

Solution First we need to know what is  $I_{\text{about } \vec{\omega} \text{ axis}} = I_{\omega}$  (say)

$$\text{let } \omega = \omega \hat{n} \quad I_{\omega} = \sum m_i |\vec{r}_i \times \hat{n}|^2 = \sum m_i |\vec{r}_i|^2 \sin^2 \theta_i$$

$$\text{also } T = \frac{1}{2} \vec{\omega} \cdot \vec{J} \quad (\text{Take origin on } \vec{\omega} \text{ axis})$$

$$\text{Now, from def}^n : \vec{J} = \sum m_i [ \vec{\omega} |\vec{r}_i|^2 - \vec{r}_i (\vec{\omega} \cdot \vec{r}_i) ]$$

$$\Rightarrow T = \frac{1}{2} \vec{\omega} \cdot \vec{J} = \frac{1}{2} \omega^2 \sum m_i [ \hat{n} \cdot \hat{n} |\vec{r}_i|^2 - (\vec{r}_i \cdot \hat{n})(\vec{r}_i \cdot \hat{n}) ]$$

$$= \frac{1}{2} \omega^2 \sum m_i [ \vec{r}_i \cdot \vec{r}_i - (\vec{r}_i \cdot \hat{n})(\vec{r}_i \cdot \hat{n}) ]$$

$$= \frac{1}{2} \omega^2 \sum m_i [ |\vec{r}_i|^2 - |\vec{r}_i|^2 \cos^2 \theta ]$$

$$= \frac{1}{2} \omega^2 \sum m_i |\vec{r}_i|^2 \sin^2 \theta$$

$$= \frac{1}{2} I_{\omega} \omega^2 \quad (+1 \text{ में यही करते थे})$$

★  $\vec{J}$  की  $\vec{\omega}$  की direction में component =  $(\vec{J} \cdot \hat{\omega}) \hat{\omega} = I_{\omega} \vec{\omega}$   
 where  $I_{\omega}$  is moment of inertia about  $\vec{\omega}$  axis.  
 i.e.  $\sum m_i |\vec{r}_i|^2 \sin^2 \theta_i$



# MECHANICS (II) OF CONTINUOUS MEDIA

- 2 main properties of continuous media :
- Elasticity
  - Viscosity
- 3 moduli  
→ Hooke's law  
→ Poisson Ratio  
→  $\gamma = 3k(1-2\nu)$   
→  $\gamma = 2\eta(1+\nu)$   
→  $\frac{1}{\gamma} = \frac{1}{9k} + \frac{1}{3\eta}$   
→ velocity profile: terminal  
→ Reynolds number

## Isotropic solid

↑ some direction

(Crystalline solids are anisotropic as they have directional properties)

- Poiseuille's Formula
- Bernoulli Theorem
- Venturi, Pitot, Torricelli

If properties are same along all directions, solid is called isotropic.

## Elastic Force :

$$F = -kx$$

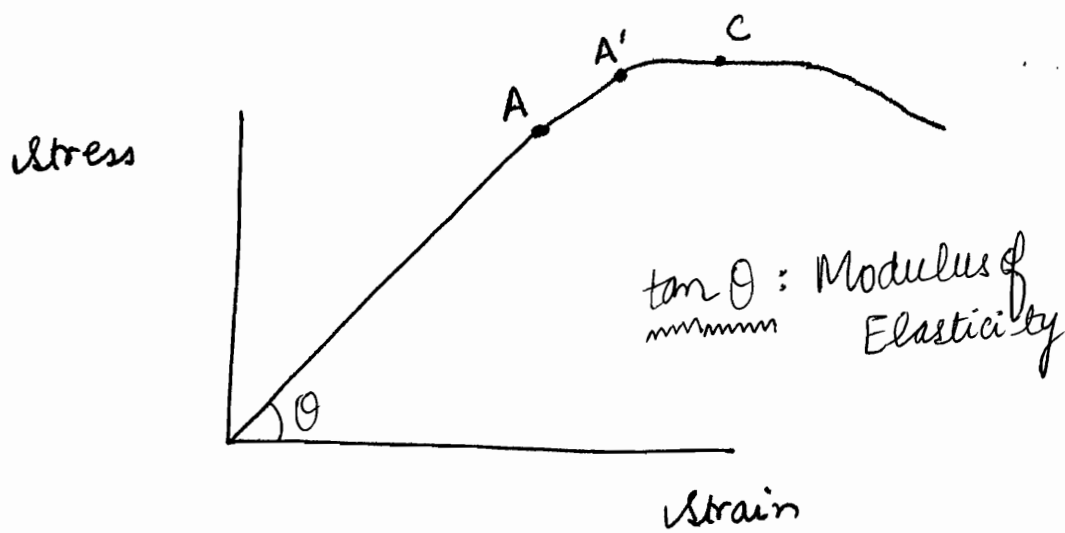
Also called restoring force.

- If elastic force same along all directions. After withdrawal of force body tries to regain shape ⇒ Elastic behaviour.
- If permanent deformation ⇒ Plastic Behaviour.
- Elastic limit: limit upto which body exhibits elastic behaviour.

• Hooke's law is valid upto Elastic limit.

## Hooke's law

Under elastic limit, stress  $\propto$  strain



A: limit of proportionality

A': limit of elasticity

Beyond A': permanent deformation

Beyond A': body flows like viscous fluid  
i.e. great strain with small stress

C: breaking stress [maximum stress]

Do not think of stress as externally applied force but rather think of it as an internal force developed within body that tries to regain shape of strained body

After that.....fatigue etc.

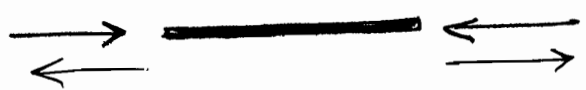
$$\left[ \frac{\text{stress}}{\text{strain}} \right] = \text{Modulus of Elasticity}$$

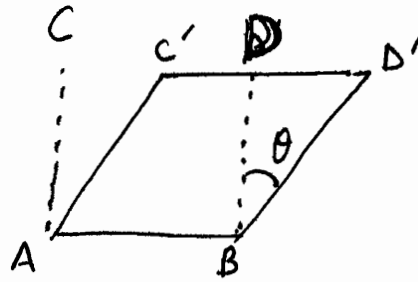
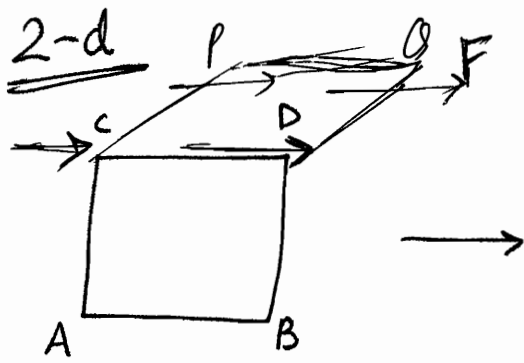
Remember  
:- Within elastic limits

↑ Note that modulus of elasticity is a general term

In 1-d

$$\left[ \frac{\text{longitudinal stress}}{\text{longitudinal strain}} \right] = \underline{\underline{\text{Young's Modulus } [\gamma]}}$$





• Tangential Stress = Modulus of Rigidity  $[\eta]$

Shear

$$= \frac{[F/A]}{\theta}$$

⊛ Here, strain is called shear

⊛ Shear strain is the relative displacement between any two planes in the body unit distance apart

✓ here A refers to area of PQDC

3-d

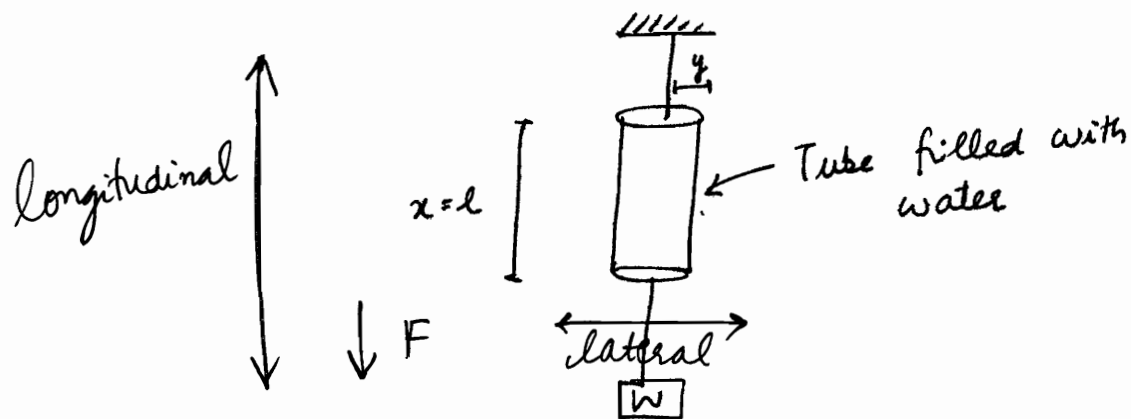
• Stress =  $\frac{dp}{-(\frac{dv}{v})}$  = Bulk Modulus

⊙  $\gamma, k, \eta$  are having dimensions of stress, and all are positive.

Within the limit of elasticity,

Poisson Ratio  $\nu = \left[ \frac{\text{lateral strain}}{\text{longitudinal strain}} \right]$  → both numerator & denominator are dimensionless

→ Tube of cycle is good example of elastic material



After application of external stress,

$x$  becomes  $(x+dx)$

$y$  becomes  $(y-dy)$

Total volume of water is constant.

$$\text{Poisson Ratio} = \nu = \frac{\left[ \frac{dy}{y} \right]}{\left[ \frac{dx}{x} \right]}$$

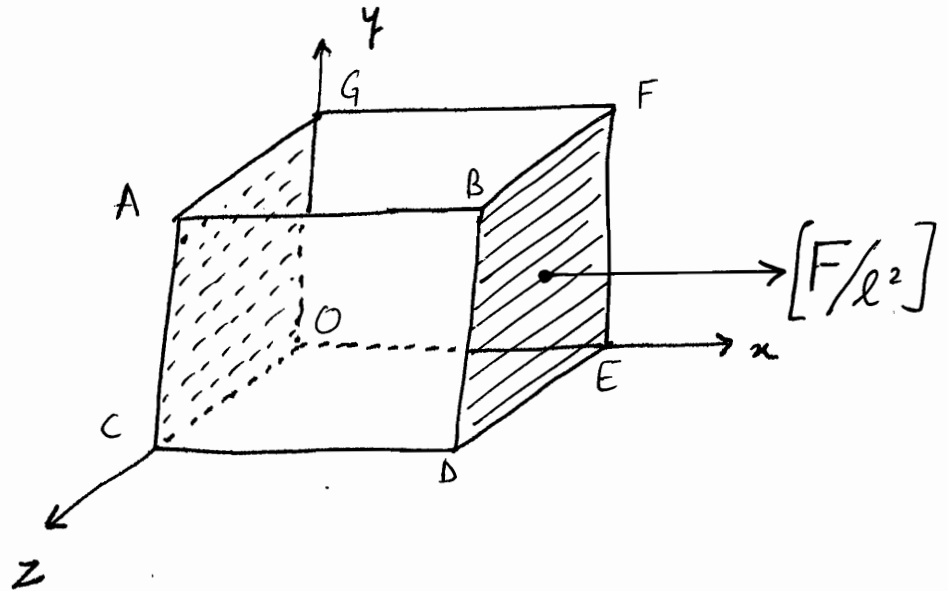
$$\begin{aligned} \bullet \text{ Lateral strain} &= \nu [\text{longitudinal strain}] \\ &= \nu \left( \frac{P}{Y} \right) \end{aligned}$$

$$\bullet \text{ lateral change in length} = \frac{\nu P}{Y} \cdot (\text{lateral length})$$

# Relationships between different Modulus of Elasticity

$$\textcircled{1} Y = 3k(1 - 2\sigma)$$

[6 faces of outside]  
along  $\vec{A}$



Initial volume =  $l^3$

Extensional stress, like  $\frac{F}{l^2}$ , applied on all faces,   
 or Tensional simultaneously

We could have applied Compressional stress also.

$$k = \frac{[F/l^2]}{[\frac{\Delta V}{V}]}$$

$$P = [F/l^2]$$

Considering P along x direction

Longitudinally  
We know

$$Y = \frac{F/l^2}{dl/l}$$

$$\Rightarrow dl_x = \frac{F}{l} \frac{l}{Y} = \left[ \frac{Pl}{Y} \right]$$

Laterally

$$dl_y = -\frac{\sigma Pl}{Y}$$

$$dl_z = -\frac{(\sigma Pl)}{Y}$$

Similarly considering all directions

$$l_x' = l + \frac{Pl}{Y} - \frac{\sigma Pl}{Y} - \frac{\sigma Pl}{Y}$$

$$l_y' = l - \frac{\sigma Pl}{Y} + \frac{Pl}{Y} - \frac{\sigma Pl}{Y}$$

$$l_z' = l - \frac{\sigma Pl}{Y} - \frac{\sigma Pl}{Y} + \frac{Pl}{Y}$$

New Volume  $V' = l_x' l_y' l_z'$

$$= \left[ l + \frac{Pl}{Y} (1-2\sigma) \right]^3$$
$$= l^3 \left[ 1 + \frac{P}{Y} (1-2\sigma) \right]^3$$

Expanding Binomially,  $V' = l^3 \left[ 1 + \frac{3P}{Y} (1-2\sigma) \right]$

$$= V + \frac{3P}{Y} (1-2\sigma) V$$

$$\Rightarrow \frac{dV}{V} = \frac{3P}{Y} (1-2\sigma)$$

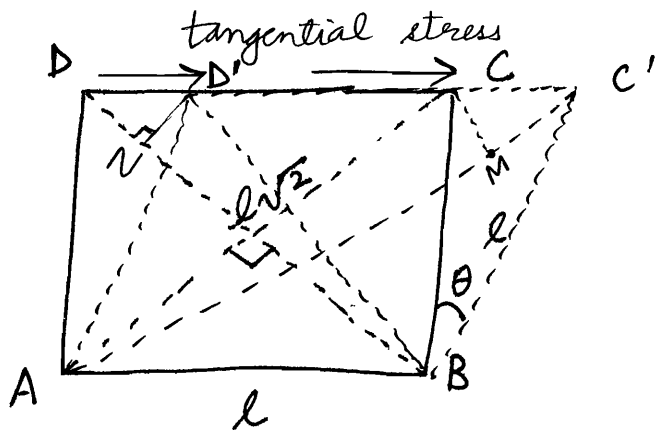
$$k = \frac{P}{\left(\frac{dV}{V}\right)} = \frac{Y}{3(1-2\sigma)}$$

$$\Rightarrow \boxed{Y = 3k(1-2\sigma)}$$

↑                    ↑                    ↑  
1-d                    3d                    2d

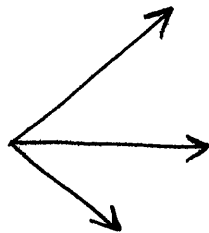


(2)



After application of tangential stress,  
 extension along AC  
 compression along BD

$$\theta_{\text{shear}} = \left[ \frac{CC'}{BC} \right]$$



Tangential stress is equivalent to extensional stress along one perpendicular direction and compressional stress along ~~the~~ other direction

$$\theta_{\text{shear}} = \left( \frac{\theta}{2} \right)_{\text{Comp.}} + \left( \frac{\theta}{2} \right)_{\text{Extens.}}$$

i.e.  $\theta = \text{ext. strain} + \text{compressional strain}$

$$AM = l\sqrt{2} = AC$$

$$\Delta AC = MC'$$

$$\textcircled{c} \text{ Extensional Shear } = \left[ \frac{MC'}{AC} \right]$$

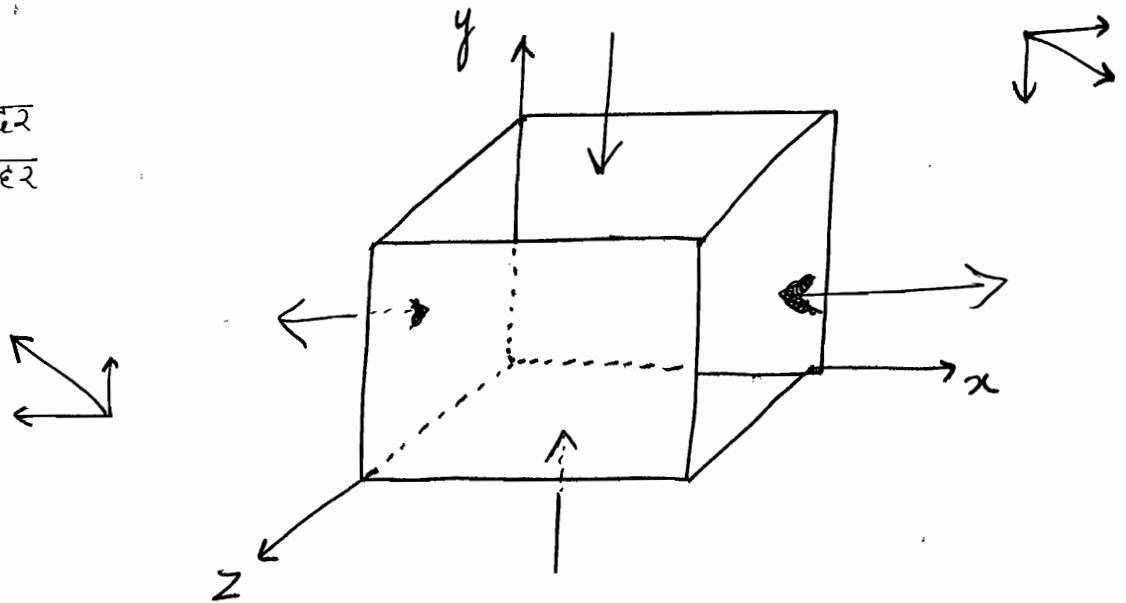
$$\Rightarrow CC' \cos 45^\circ = CM$$

$$\frac{CC'}{\sqrt{2}} = CM$$

$$\begin{aligned}
 &= \frac{MC'}{AC} \\
 &= \frac{CC'}{\sqrt{2}(\sqrt{2}BC)} \\
 &= \frac{1}{2} \cdot \left[ \frac{CC'}{BC} \right] \\
 &= \frac{1}{2} [\theta_{\text{shear}}]
 \end{aligned}$$

◉ Compressional shear =  $\left( \frac{DN}{DB} \right) = \frac{DD' \cos 45^\circ}{AD \sqrt{2}} = \frac{1}{2} \left[ \frac{DD'}{AD} \right]$   
 =  $\left[ \frac{\theta_{\text{shear}}}{2} \right]$  similarly

- 2 faces पे अंदर
- 2 faces पे बाहर
- 2 faces पे 0



Tangential stress along z-axis  
 = Compression stress along y axis  
 +  
 Extensional stress along x-axis

$$l_{x'} = l + \frac{Pl}{Y}$$

$$l_{y'} = l - \left[ \frac{\sigma Pl}{Y} \right]$$

$$l_z = l - \left[ \frac{\sigma Pl}{Y} \right]$$

due to extensional  
stress along  
x-axis

[Note that the two signs will  
be same]

$$l_{x'} = \left( l + \frac{Pl}{Y} \right) + \frac{\sigma Pl}{Y}$$

$$l_y = \left( l - \frac{\sigma Pl}{Y} \right) - \frac{Pl}{Y}$$

$$l_z = \left( l - \frac{\sigma Pl}{Y} \right) + \frac{\sigma Pl}{Y}$$

due to compressional  
stress along y

NOTE THAT ALL FORCES ARE ACTING SIMULTANEOUSLY

$$l_{x'} = l + \frac{Pl}{Y} (1 + \sigma)$$

$$l_{y'} = l - \frac{Pl}{Y} (1 + \sigma)$$

$$\text{Extensional shear} = \frac{\theta}{2} = \frac{P}{Y} (1 + \sigma)$$

$$\text{Compressional shear} = \frac{\theta}{2} = \frac{P}{Y} (1 + \sigma)$$

$$\text{Tangential shear} = \frac{2P}{Y} (1 + \sigma)$$

$$\boxed{1/\eta = \frac{2}{Y} (1 + \sigma)} \Rightarrow$$

$$Y = 2\eta(1 + \sigma)$$

Eliminating  $\sigma$  from 2 expression

$$Y = 3k(1 - 2\sigma) = 2\eta(1 + \sigma)$$

$$\circ \sigma < \frac{1}{2}$$

$$\circ \sigma > -1$$

$$\Rightarrow -1 < \sigma < \frac{1}{2}$$

Ordinarily,  $0.2 < \sigma < 0.4$

$$Y = 3k \left[ 1 - 2 \left( \frac{Y}{2\eta} - 1 \right) \right]$$

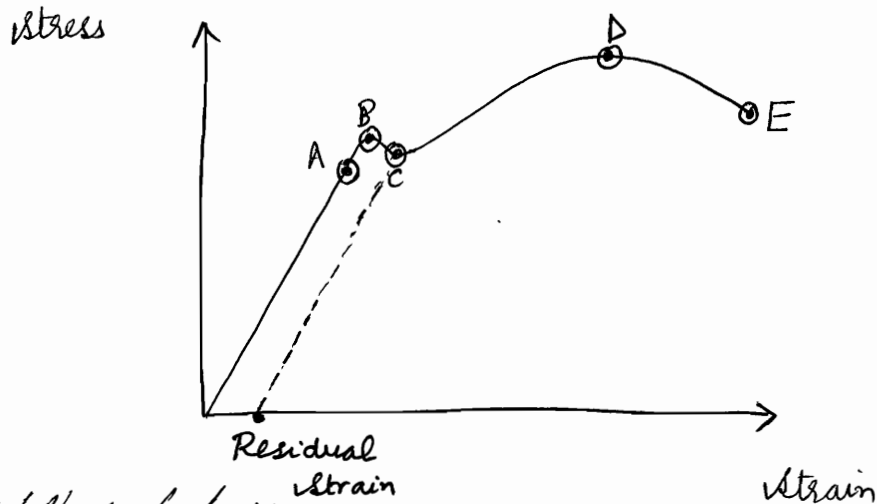
$$= 3k \left[ 3 - \frac{Y}{\eta} \right]$$

$$Y \left[ 1 + \frac{3k}{\eta} \right] = 9k$$

$$\Rightarrow Y = \frac{9\eta k}{\eta + 3k}$$

$$\Rightarrow \frac{1}{Y} = \frac{1}{9k} + \frac{1}{3\eta}$$

## Relationship between Stress and Strain



~~Elastic limit  
Yield point  
Permanent strain  
Ultimate stress  
Breaking stress~~

A: Proportional limit

B: Elastic limit or Yield Point

C: Permanent strain

D: Ultimate Tensile Strength

E: Fracture Point

⊛ Work done by the deforming force per unit volume  $= \frac{1}{2} * \text{stress} * \text{strain}$

⊛ Results that connect  $\nu$ ,  $\eta$  and  $k$  may not be applicable to materials in the form of a wire, because in the process of being drawn out, the outer layers of the wires become invariably harder than the inner ones and their elastic properties are then considerably altered.

① In elasticity experiments, we usually use two materials of same type. 1 acts as "control" apparatus i.e. the effects of the changes other than applied eg. Temp, Pressure can be accounted for by studying changes in the unused "control" wire.

Q. A uniform tube  $l$  m long and closed at bottom end is completely filled with water. Upper end is clamped & is stretched downward. It is found that length of tube increases by  $l_t$  but that of water column increases by  $l_w$ . Obtain Poisson ratio

Volume of water remains same

$$\pi r^2 l = \pi (r - dr)^2 (l + dl_w) \quad \text{--- (1)}$$

$$\text{Also } \sigma = \frac{\left(\frac{dr}{r}\right)}{\left(\frac{dl_w}{l}\right)} \Rightarrow \boxed{\left(\frac{dr}{r}\right) = \sigma \left(\frac{dl_w}{l}\right)} \quad \text{--- (2)}$$

Using (2) in (1)

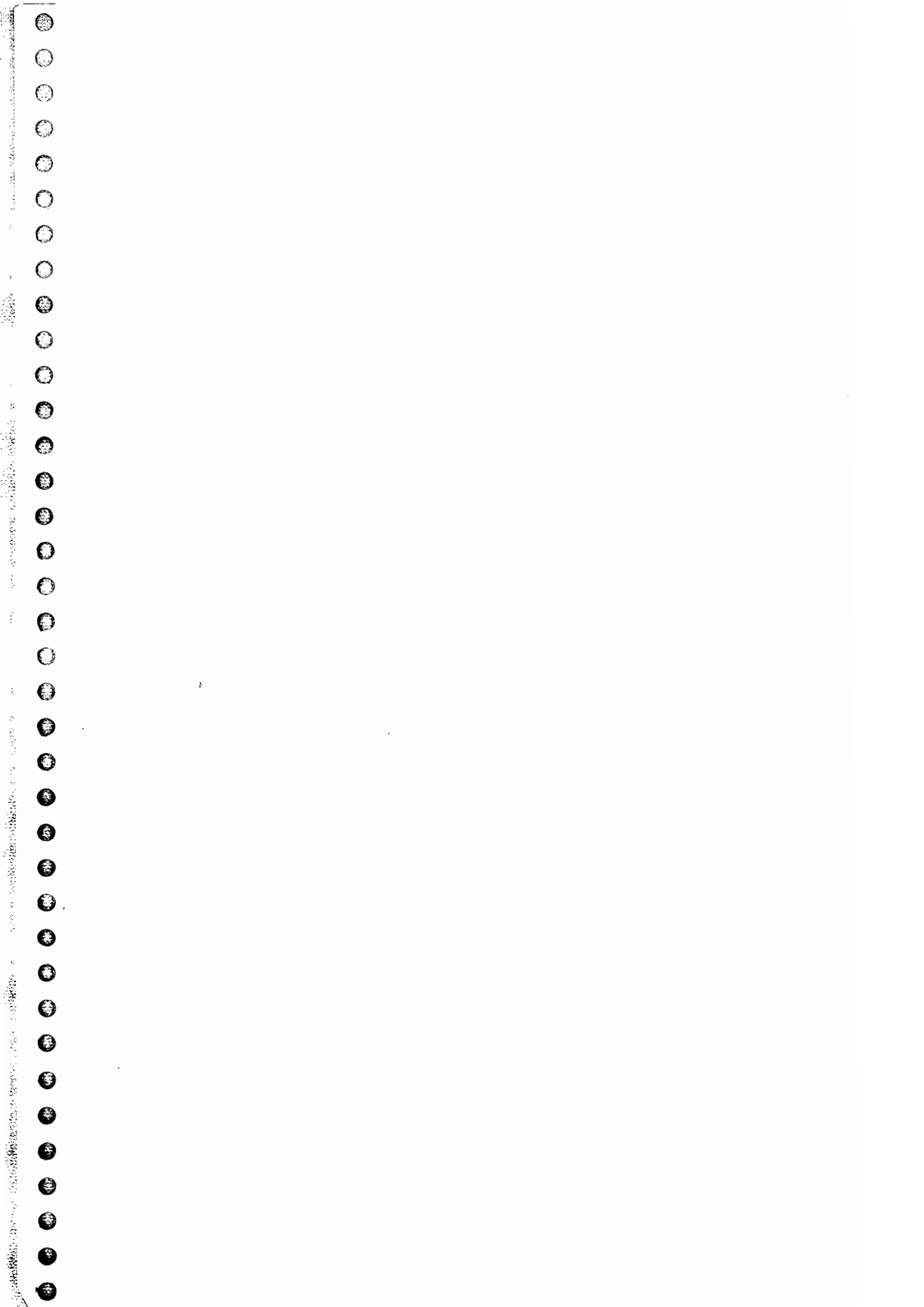
$$1 = \left(1 - \frac{dr}{r}\right)^2 \left(1 + \frac{dl_w}{l}\right)$$

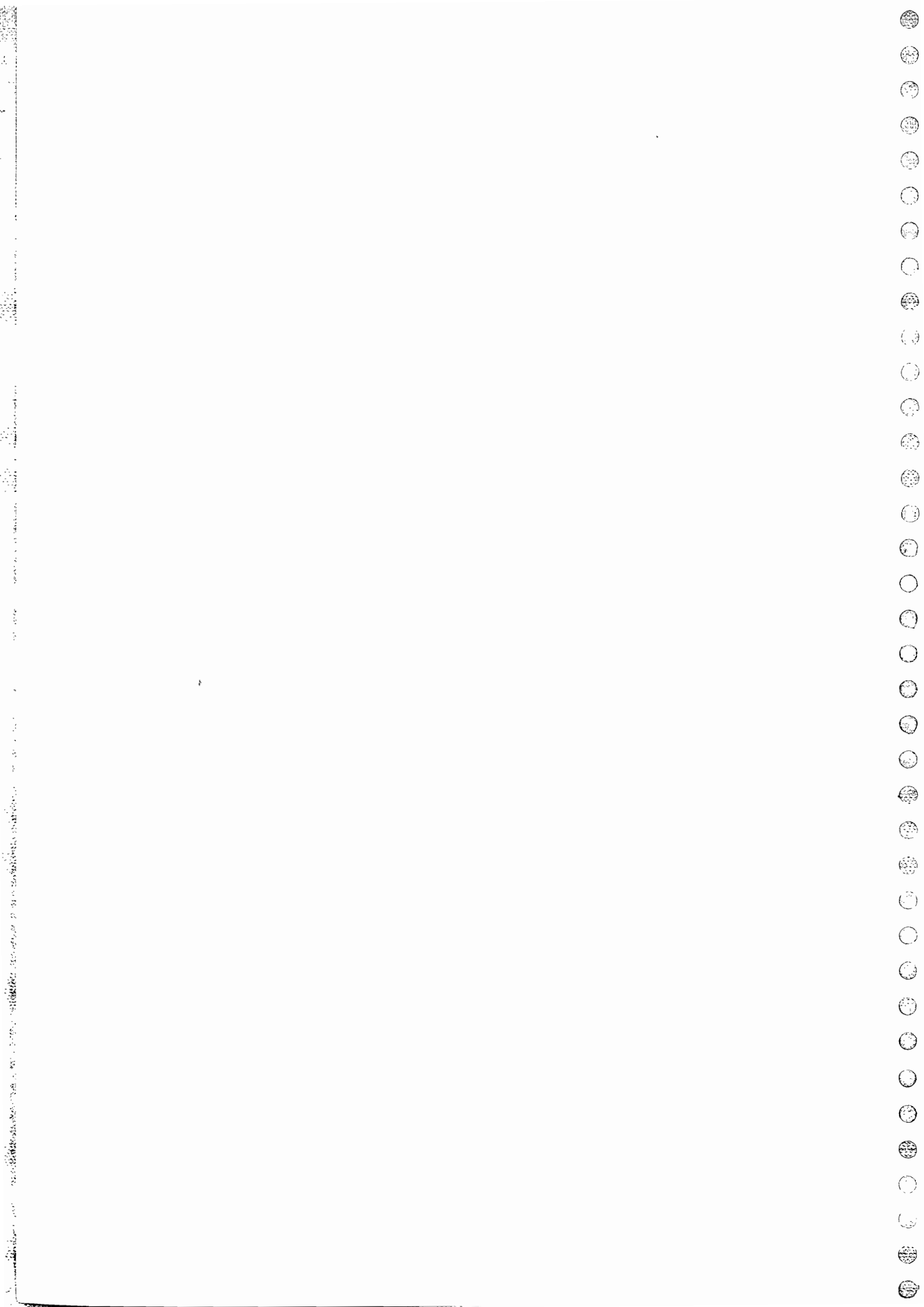
$$\Rightarrow 1 = \left[1 - \sigma \left(\frac{dl_w}{l}\right)\right]^2 \left[1 + \frac{dl_w}{l}\right]$$

$$\Rightarrow \left[1 - \sigma \left(\frac{dl_w}{l}\right)\right]^{-2} = 1 + \frac{dl_w}{l}$$

$$\Rightarrow 1 + 2\sigma \left(\frac{dl_w}{l}\right) = 1 + \left(\frac{dl_w}{l}\right)$$

$$\Rightarrow \sigma = \frac{\left(\frac{dl_w}{l}\right)}{2 \left(\frac{dl_w}{l}\right)} = \frac{l_w}{2 l_t} \quad \checkmark$$







# MECHANICS (12)

## VISCOSITY

Stokes law & application :

$$F = 6\pi \eta r v \quad (\text{only for spherical body, damping force})$$

Medium resistance is dependent upon  $\vec{v} \Rightarrow$  non conservative force.

✓ Viscosity is resistance offered to motion.

It has been found

Resistance  $\propto |\vec{v}|$


Viscosity is behaviour of fluids (liquids & gases) by which they oppose the motion

$$F \propto A$$

$$F \propto \frac{dv}{dz}$$

$$F = -\eta A \frac{dv}{dz}$$

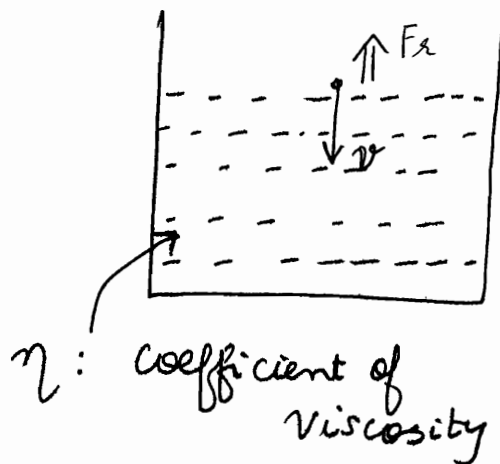
Newton's  
Formula

$\Rightarrow$    
note the  
minus sign  
This is not due to  
direction ... but due to  
fact that  $\left(\frac{dv}{dz}\right)$  is negative !!

## General

$$\text{Force in air} = -b\vec{v}$$

↑  
damping  
constant



Now if the shape is spherical,

$$F \propto \vec{v}$$

$$F \propto r\vec{v}$$

$$F \propto \eta r\vec{v}$$

empirical formula

$$F = c \cdot \eta r\vec{v}$$

$c = 6\pi$  found experimentally

$$\Rightarrow \boxed{\vec{F} = -6\pi\eta r\vec{v}}$$



## Terminal velocity

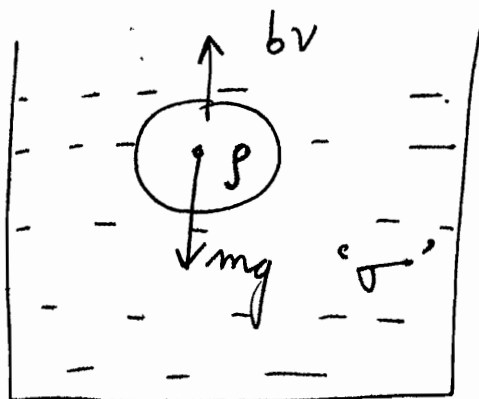
Also include upthrust, if not small.

$$m \left( \frac{d\vec{v}}{dt} \right) = F_{\text{net}} = -b\vec{v} + mg\vec{j}$$

@ Terminal velocity,  $F_{\text{net}} = 0$

$$m \frac{dv}{dt} = 0 \Rightarrow v = \text{const.}$$

$$v_T = \left( \frac{mg}{b} \right)$$



$$m \frac{dv}{dt} + bv = mg$$



velocity Profile

$$\int \frac{dv}{mg - bv} = \int \frac{dt}{m}$$

Do not forget 'g' in the formula

$$v_T = \frac{2r^2 g (\rho - \sigma)}{9\eta}$$

Upthrust = Amount of liq. displaced  
 $= \frac{4}{3} \pi r^3 \sigma g$

$$m \frac{dv}{dt} = \frac{4}{3} \pi r^3 \sigma g + 6\pi \eta r v - \frac{4}{3} \pi r^3 \rho g$$

Ball can either go up or down depending upon densities

@ terminal velocity,  $\frac{dv}{dt} = 0$

$$\frac{4}{3} \pi r^3 g (\sigma - \rho) = - \frac{3}{6} \pi \eta r v_T$$

$$\Rightarrow \frac{2r^2 g (\rho - \sigma)}{9\eta} = v_T$$

$\rho > \sigma$  : falling down

$\sigma > \rho$  : rising

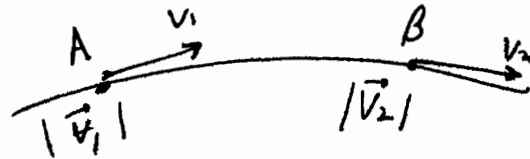
2<sup>nd</sup> type of question is :  $v = f(t) + k$   
 Then find t when  $v = \left(\frac{v_T}{e}\right)$

take care of definite integral

$$v = \frac{2r^2 g (\rho - \sigma)}{9\eta} \left(1 - e^{-\left(\frac{9\eta t}{2r^2 g}\right)}\right)$$

## Streamline Flow or laminar flow:

If all particles passing through a particular point have same velocity  $\Rightarrow$  streamline flow



$$\left[ \frac{\text{Inertial force}}{\text{Viscous force}} \right] = \text{Reynolds Number } R.$$

R is characteristic of Flow.

"drive"

$$R = \left[ \frac{\rho v D}{\eta} \right]$$

D : diameter of pipe

If,  $R < \underline{2000}$  : laminar flow  
 $R > 3000$  : turbulent flow  
 $2000 < R < 3000$  : transition region

: Open channels

$R < \underline{1600}$  : laminar flow

: narrow pipes  
or  
capillaries

$$\frac{\rho v_c D}{\eta} = 1000$$

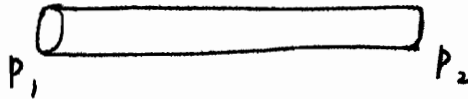
$$v_c = \left[ \frac{1000 \eta}{\rho D} \right]$$

Critical velocity

{ always mention the assumptions associated with the capillary tube }

### Poiseuille's Formula

It deals with pipes of uniform cross section <sup>①</sup> ~~uniform cross section~~ and flow is streamline <sup>②</sup>. Incompressible fluid. <sup>③</sup>



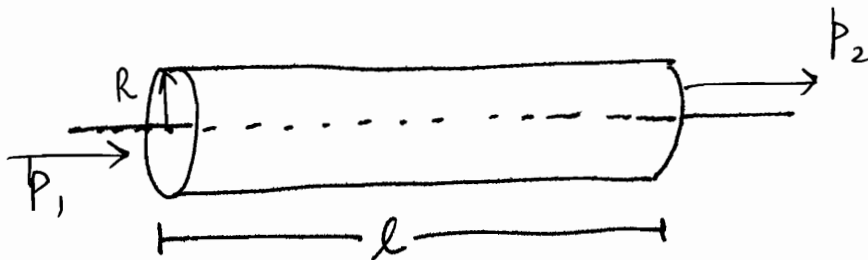
③  $P_1 - P_2 = \text{const.}$

$$\text{Flow} : \frac{Q \text{ i.e. (volume of water)}}{t} = \frac{d(\text{Vol})}{dt} = \frac{dx \cdot \text{Area}}{dt}$$

$$= \vec{v} \cdot \vec{A}$$

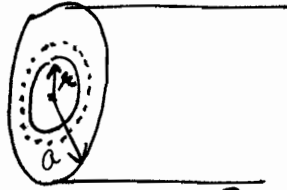
$$= vA$$

Flow is measured in cusecs i.e. m<sup>3</sup>/sec



there is no radial flow, <sup>④</sup> only longitudinal flow due to const [P<sub>1</sub> - P<sub>2</sub>]. All fluid particles in contact with walls have velocity = 0 <sup>⑤</sup>

$$F = (p_1 - p_2) A$$



{ at equilibrium

{ Force pushing the fluid = Force resisting the flow }

$$F = -\eta A \frac{dv}{dx}$$

Remember the 'minus' sign  
 $\therefore \left(\frac{dv}{dx}\right)$  is negative  
 not due to direction

$$(p_1 - p_2) \pi r^2 = -\eta 2\pi r l \left(\frac{dv}{dx}\right)$$

$$dv = -\frac{p}{4\eta l} \int r \, dr$$

$$v = -\frac{p}{4\eta l} r^2 + A_{\text{const.}}$$

directly do definite integral

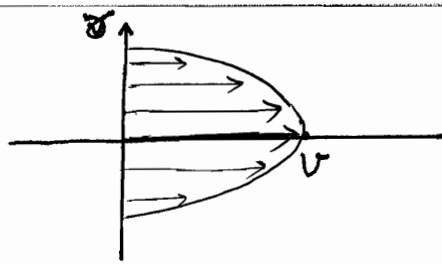
$$\int_r^0 = \int_r^a$$

At  $r = a$ ,  $v = 0$

$$\Rightarrow A_{\text{const.}} = \frac{p}{4\eta l} a^2$$

$$\Rightarrow v = \frac{p}{4\eta l} [a^2 - r^2]$$

Can be derived



Parabolic  
velocity profile

$$dQ = v \, dA = \frac{P}{4\eta l} [a^2 - r^2] \cdot 2\pi r \, dr$$

$$= \frac{\pi P}{2\eta l} (a^2 r - r^3) \, dr$$

$$Q = \int dQ = \frac{\pi P}{2\eta l} \int_0^a (a^2 r - r^3) \, dr$$

$$Q = \frac{\pi P a^4}{8\eta l}$$

Remember !!

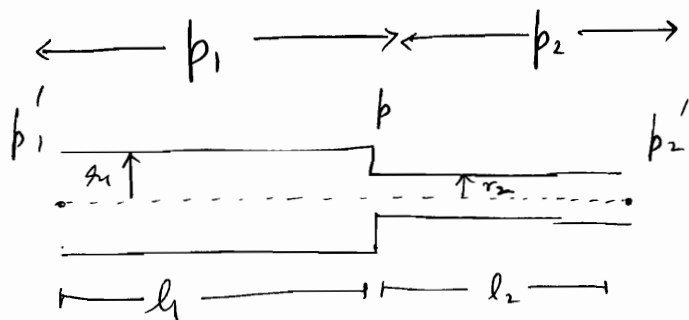
### Series Combination

$$p_1' - p_2 = \text{const} = p_1$$

$$p - p_2' = \text{const} = p_2$$

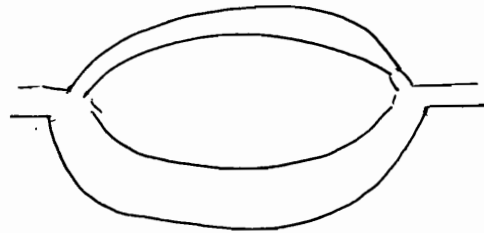
$$p_1' - p_2' = [p_1 + p_2]$$

Since series  $Q_1 = Q_2$



$$\frac{\pi P_1 r_1^4}{8 \eta l_1} = \frac{\pi P_2 r_2^4}{8 \eta l_2}$$

## Parallel Combination



$$Q = Q_1 + Q_2$$

$$P_1 = P_2$$

## Q2) Tut 5

$$\frac{P_1 r^4}{l} = \frac{P_2 \left(\frac{r}{2}\right)^4}{2l} = \frac{P_3 \left(\frac{r}{3}\right)^4}{\left(\frac{l}{2}\right)}$$

$$\Rightarrow \underline{P_2 = 32 P_1}$$

$$\underline{P_3 = \frac{81}{2} P_1}$$

## Bernoulli's Theorem

① For any ideal fluid

- ① non viscous
  - ② incompressible
  - ③ streamline
- Conservative  
Continuity  
flow along this line

sum of k.E, P.E. and Pressure Energy per unit volume remains const.



- Pressure Energy per unit volume =  $P$
- K.E. per unit volume =  $\frac{1}{2} \rho v^2$
- P.E. per unit volume =  $\rho gh$

$$P + \frac{1}{2} \rho v^2 + \rho gh = \text{const.}$$

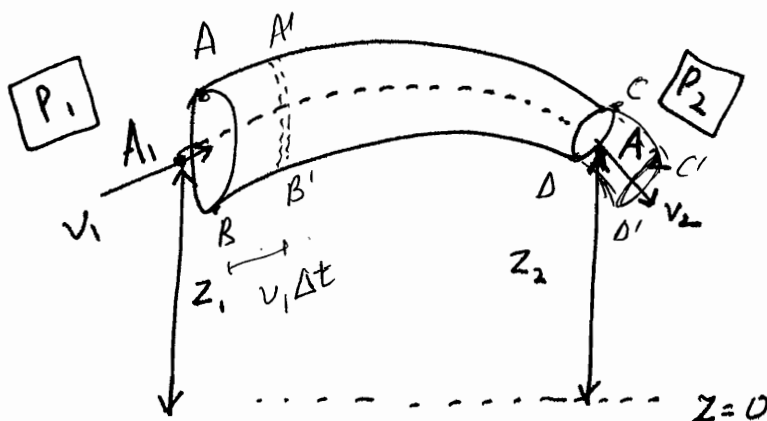
When expressed in terms of length, the 3 units are called heads.

$$\frac{P}{\rho g} + \frac{1}{2} \left( \frac{v^2}{g} \right) + h = \text{const.}$$

$\uparrow$  Pressure head       $\uparrow$  velocity head       $\uparrow$  Potential head

Since ideal fluid taken  $\Rightarrow$  Conservative forces

$$\Rightarrow \underline{W_{\text{done}} = \Delta \text{K.E.} + \Delta \text{P.E.} = \Delta (\text{Mechanical Energy})}$$



$$\left[ P_1 + \frac{1}{2} \rho v_1^2 + \rho z_1 g = P_2 + \frac{1}{2} \rho v_2^2 + \rho z_2 g \right] \text{ derivation}$$



$$W = F dx$$

$$= P_1 A_1 v_1 \Delta t$$

On other side

$$W = - P_2 A_2 v_2 \Delta t$$

$$\Rightarrow \text{Net Work done} = \frac{P_1 A_1 v_1 \Delta t - P_2 A_2 v_2 \Delta t}{}$$

Since ideal fluid, incompressible fluid

$$\Delta m = \rho A_1 v_1 \Delta t = \rho A_2 v_2 \Delta t$$

$$\Rightarrow \frac{A_1 v_1}{} = A_2 v_2$$

Continuity eqn

$$\Rightarrow \text{Work Done} = \frac{\Delta m}{\rho} (P_1 - P_2)$$

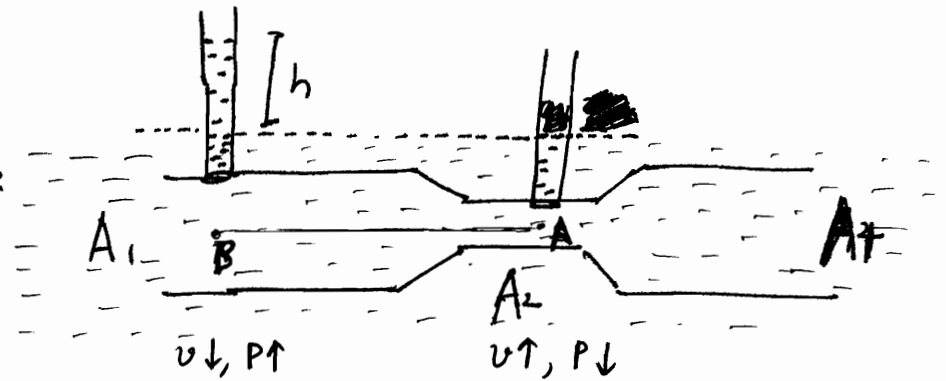
$$\text{Work Done} = \Delta ME = \frac{1}{2} \Delta m (v_2^2 - v_1^2) + \Delta m (z_2 - z_1) g$$

$$\Rightarrow \frac{P_1}{\rho} + \frac{v_1^2}{2} + z_1 g = \frac{P_2}{\rho} + \frac{v_2^2}{2} + z_2 g$$

$$\Rightarrow P + \frac{1}{2} \rho v^2 + \rho g z = \text{const.}$$

## Applications

1) Venturimeter :



$$Q = C \sqrt{h}$$

$$A_1 v_1 = A_2 v_2$$

$$P_A + \frac{1}{2} \rho v_2^2 = P_B + \frac{1}{2} \rho v_1^2$$

$$\frac{1}{2} \rho (v_2^2 - v_1^2) = P_B - P_A$$

$$\frac{1}{2} \rho v_2^2 \left[ 1 - \frac{A_2^2}{A_1^2} \right] = P_B - P_A = \rho g h$$

$$v_2^2 = \frac{\rho g h \frac{2 A_1^2}{\rho [A_1^2 - A_2^2]}}{\left[ \frac{2 A_1^2 g}{A_1^2 - A_2^2} \right] h}$$

$$\begin{aligned} \text{Flow} = v_2 \cdot A_2 &= \sqrt{v_2^2 A_2^2} \\ &= \sqrt{\frac{2 A_1^2 A_2^2 g}{A_1^2 - A_2^2}} \cdot h \end{aligned}$$

$C_1$ : Correction  
const.

$$= C_1 \sqrt{\frac{2 A_1^2 A_2^2 g}{(A_1^2 - A_2^2)}} \sqrt{h}$$

$$Q = C \sqrt{h}$$

$C$ : Calibration  
const.

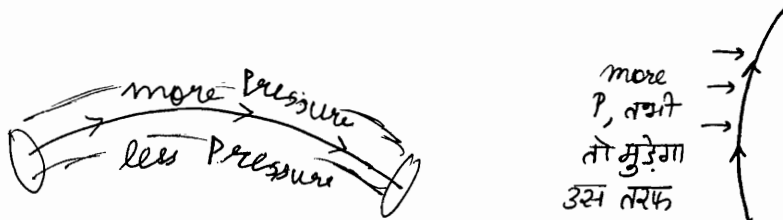
# Streamline Flow

The flow of the fluid is said to be streamline or laminar flow, if the velocity at every point in the fluid remains const, in magnitude as well as direction.

~~The energy needed to drive the fluid being used up in overcoming the 'viscous drag' between the layers.~~

We define 'streamline' as a curve, the tangent to which ~~at~~ any point gives the direction of fluid flow at that point.

A streamline may be straight (equal pressure on both sides) or curved (Pressure more on convex side)



Fluid flow remains ~~constant~~ streamline till velocity does not exceed a limiting value called its critical velocity, beyond which flow becomes turbulent.

⊛ Remember  $Q$  is rate of flow of fluid  
 $Q = \left(\frac{dV}{dt}\right) = Av$   
→  $v$ : velocity  
→  $V$ : Volume

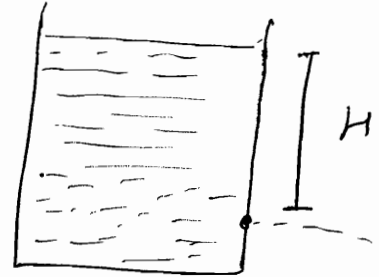
⊛ In order to visualize Pressure energy, imagine the velocity of efflux, release of Pressure gives rise to Kinetic Energy.

⊛ Bernoulli Theorem states that the total energy of an incompressible fluid, non-viscous fluid in steady flow remains constant throughout the flow. Total energy comprises kinetic Energy, Potential Energy and Pressure Energy.

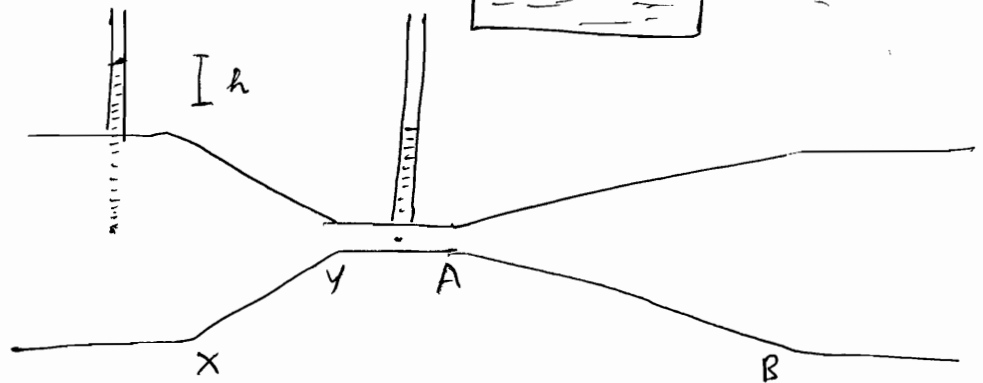
## Tollicelli theorem and/or law of efflux

The velocity of efflux of a liquid issuing out of an orifice is the same as it would attain if allowed to fall freely through the vertical height between the liquid surface and orifice.

$$v = \sqrt{2gH}$$



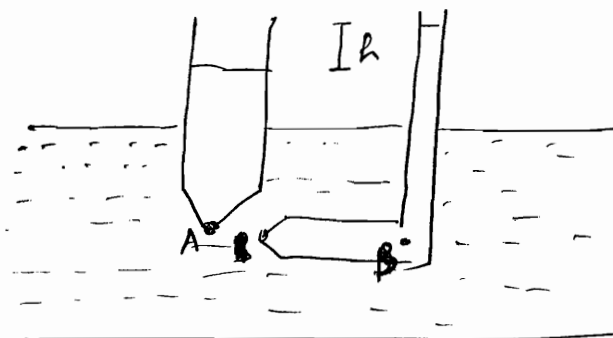
## Venturimeter



There is no harm in hastening the acceleration of the stream, but hastening its deceleration ~~leads~~ leads to "Boundary Layer Effect" i.e. changes in the flow pattern along boundaries, and the flow no longer remains streamline. This is the reason for gentle slope AB.

$$\text{Throat ratio} = \left(\frac{r}{R}\right) = \left(\frac{\text{radius of narrow tube}}{\text{RADIUS of broad tube}}\right) < 1$$

## Pitot Tube



Note that in Bernoulli, we consider non-viscous flow i.e. velocity is same across the cross section



For measure of velocity of fluid and corresponding flow. The open L-shaped tube is called Pitot tube

$$P_A + \frac{1}{2} \rho v^2 = P_B \Rightarrow v = \sqrt{2gh}$$

- ⊛ In Turbulent flow, most of the energy needed to drive the liquid is now dissipated in setting up eddies & whirlpools in it.

$$v_c = \left( \frac{\eta R}{\rho D} \right) \quad R: \text{Reynold's Number}$$

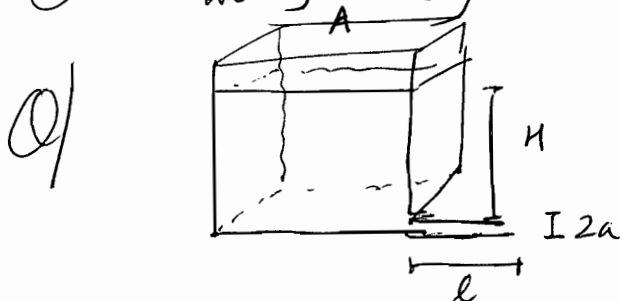
Reynold's Number is, in fact, ratio of inertia and viscous forces. RAYLEIGH pointed out that two fluids - flows or motions of two bodies immersed in viscous media are dynamically or mechanically equivalent if

- (a) Shapes of the pipes or those of moving bodies are geometrically similar, whatever their actual dimensions.

and (b) The Reynold's number is same for both.

### Limitation of Poiseuille's Formula

- ① Not valid for high velocities or flows
- ② Assumes incompressible fluid, streamlined flow & no radial flow
- ③ Velocity along the ~~radius~~ <sup>tube</sup> = 0, assumed.



Time when  $(H/2)$ .  
 $\eta$

d/

$$\frac{dv}{dt} = \frac{\pi \rho a^4}{8 \eta l} \cdot \frac{A}{dt} = \frac{\pi \rho g h a^4}{8 \eta l}$$

$$\Rightarrow -\left(\frac{dh}{h}\right) = \left(\frac{\pi \rho g a^4}{8 \eta l}\right) dt$$

$$\Rightarrow \ln\left(\frac{H_0}{H}\right) = \frac{\pi \rho g a^4 t}{8 \eta l} \Rightarrow \underline{\underline{t = \frac{8 \eta l}{\pi \rho g a^4} \ln(2)}}$$

# MECHANICS (13)

- Postulate
- Null Result
- Lorentz Transform
- Simultaneity
- Length Contraction

29/11/11

• SPECIAL THEORY OF RELATIVITY (1905) (STR)

∴ deals with only inertial frame of reference

• General Theory of Relativity (1915) (GTR) : valid for non-inertial frames

→ We are only interested in inertial frames i.e. Special Theory of Relativity only.

Michelson Morley Experiment [refer last page of copy]

## Postulates of STR

✓ ① All physical laws retain their form in all inertial frames.

eg  $x_1 = (x - vt)$  in inertial frame 1

$x_2$  may be  $\alpha(x - vt)$  in inertial frame 2

it can be like this but not dependence upon, say  $t^2$ .

→ Tensor :- transformations from 1 frame to other ~~frame~~  
Equations frame.

✓ ② velocity of light in free space is invariant in all inertial frames.

(Nothing betrays uniform motions, not even light)

< Phenomenon >  
- scope : detection device  
- meter : measuring device

⊙ Postulate 2 was a result of Michelson Morley Experiment if  $(c-v)$ ,  $(c+v)$ , there would have been fringe shifts in interference of light. But no fringe shift was observed, i.e. Null result.

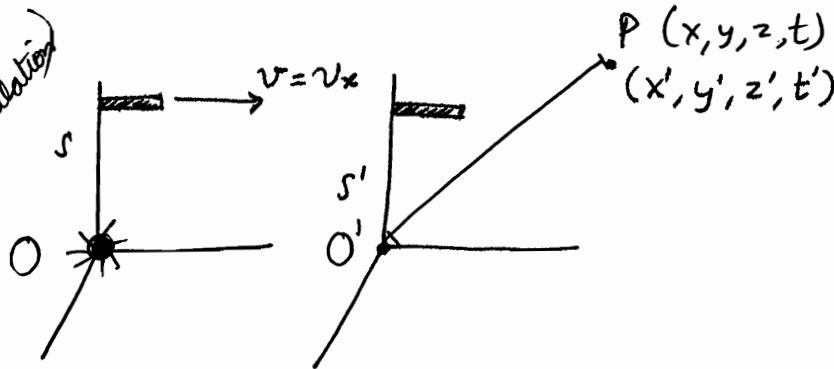
This led Einstein to deduce that  $c$  is constant.

< refer to Null result @ end of notes >

Lorentz Transformation [going from 1 inertial frame to other by applying postulates of STR]

Both the observers coincide at  $t=0$ .  $S$ : fixed observed  
 $S'$ : moving with  $\vec{v}$  along  $x$ -axis.

Theory + 5 step derivation (no need to show calculation)



Some events occurs at point P

No velocity along  $y$  or  $z$ .  $\Rightarrow$  NO coordinate change along  $y$  and  $z$ .

$$\Rightarrow y' = y$$

$$\Rightarrow z' = z$$

Galilean Transforms

$$x - x' = vt$$

$$\Rightarrow x' = x - vt$$

$$\Rightarrow t' = t$$

[suppose a light source is situated at origin O in frame S, from which a wavefront of light is emitted at  $t=0$ . When the light reaches at point P, let the position & time measured by O and O' be  $(x, y, z, t)$  &  $(x', y', z', t')$  ★

Now if event is related to say flashing a light.

We know from Postulate 2 that  $c_1 = c_2 = c$



$$OP = \sqrt{x^2 + y^2 + z^2} = ct$$

$$O'P = \sqrt{x'^2 + y'^2 + z'^2} = ct'$$

According to STR, velocity of light will be  $c$  in both the frames. Hence

$$t = \frac{OP}{c} = \frac{(x^2 + y^2 + z^2)^{1/2}}{c}$$

$$t' = \frac{O'P}{c} = \frac{(x'^2 + y'^2 + z'^2)^{1/2}}{c}$$

if  $t = t' \Rightarrow x = x' = \text{contradiction}$

$$\Rightarrow \underline{t \neq t'}$$

Let,

$$t' = \beta t + \gamma x$$

$$x' = d(x - vt)$$

} step 1

(1<sup>st</sup> law of STR)

Lorentz transforms

$$y' = y$$

$$z' = z$$

⊙ We note that for observer  $O$ , distance  $OO' = vt$  &  $\therefore$  when  $x' = 0$  (point  $O'$ ),  $x = vt$ . This suggests,  $x'$  must be of form  $\alpha(x - vt)$ , b'coz only then, for  $x' = 0$ ,  $x = vt$ .

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = 0$$

$$x^2 + y^2 + z^2 - c^2 t^2 = 0$$

} invariance of  $[x_i^2 + y_i^2 + z_i^2 - c^2 t_i^2]$

in  $i$  frame where

$i$  frame  $\in$  {all frames} inertial

$$x^2 - c^2 t^2 = x'^2 - c^2 t'^2$$

} step 2 (2<sup>nd</sup> law of STR)

$$x^2 - c^2 t^2 = d^2 [x^2 + v^2 t^2 - 2xvt] - c^2 [\beta^2 t^2 + \gamma^2 x^2 + 2\beta \gamma xt]$$

Equating coefficients } step 3

$$1 = d^2 + c^2 \gamma^2$$

$$-2d^2 v - 2c^2 \beta \gamma = 0$$

$$-c^2 = d^2 v^2 - c^2 \beta^2$$

$$\alpha^2 - \gamma^2 c^2 = 1 \quad \text{--- (1) } \dots \alpha > 0$$

$$\alpha^2 v^2 - \beta^2 c^2 = -c^2 \quad \text{--- (2) assume } \beta > 0$$

$$-\alpha^2 v - \beta \gamma c^2 = 0 \quad \text{--- (3) } \dots \gamma < 0$$

★ Note that  $\beta \gamma$  is negative. We assume  $\beta$  is +ve,  $\gamma$  is -ve

Solving 3 equations

↑  
→ refer Mathur how  $\beta t + \gamma x$  came

From (1) 
$$\alpha^2 = 1 + \gamma^2 c^2$$

From (2) 
$$\beta^2 = \frac{\alpha^2 v^2 + c^2}{c^2} = \frac{(1 + \gamma^2 c^2) v^2 + c^2}{c^2}$$

$$\beta^2 = \left[ \frac{v^2}{c^2} + \gamma^2 v^2 + 1 \right]$$

Squaring (3)



$$\alpha^4 v^2 = \beta^2 \gamma^2 c^4$$

← [remember to square the 3<sup>rd</sup> equation]

$$(1 + \gamma^2 c^2)^2 v^2 = \frac{v^2}{c^2} \cdot \gamma^2 c^4 + \gamma^2 v^2 \gamma^2 c^4 + \gamma^2 c^4$$

$$\Rightarrow v^2 + \cancel{\gamma^4 c^4 v^2} + 2\gamma^2 c^2 v^2 = v^2 \gamma^2 c^2 + \cancel{\gamma^4 v^2 c^4} + \gamma^2 c^4$$

$\gamma^4$  will get cancelled...

$$\gamma^2 c^2 v^2 - \gamma^2 c^4 = -v^2$$

$$\gamma^2 = \frac{-v^2}{-c^2 v^2 + c^4}$$

$$\gamma = \frac{-v}{\sqrt{-c^2 v^2 + c^4}} = \frac{-v}{c^2 \sqrt{1 - \frac{v^2}{c^2}}} = \frac{-v \alpha}{c^2}$$

★ @ initial take  $\beta t - \gamma x$

$$\alpha' = \sqrt{1 + \frac{v^2}{c^2(1 - \frac{v^2}{c^2})}}$$

$$= \sqrt{1 + \frac{v^2}{c^2 - v^2}} = \sqrt{\frac{c^2}{c^2 - v^2}} = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}}$$

$$\beta^2 = \frac{v^2}{c^2} + \frac{v^2 \cdot v^2}{c^4(1 - \frac{v^2}{c^2})} + 1 = \frac{v^2}{c^2} + 1 + \frac{v^4}{c^4 - v^2 c^2}$$

$$= \frac{c^2 v^2 - v^4 + c^2(c^2 - v^2) + v^4}{c^2(c^2 - v^2)} = \frac{c^2}{c^2 - v^2} = \frac{1}{1 - \frac{v^2}{c^2}} = \alpha^2$$

$$\Rightarrow \boxed{\beta = \alpha}$$

$$\Rightarrow \left\{ \begin{array}{l} \text{Step 4:} \\ \text{Write} \\ \text{down} \\ \text{values of} \\ \alpha, \beta, \gamma \end{array} \right. \left\{ \begin{array}{l} \text{Step 5} \\ \alpha' = \frac{x - vt}{\sqrt{1 - (v^2/c^2)}} \\ y' = y \\ z' = z \\ t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - (v^2/c^2)}} \end{array} \right.$$

Note that  $\alpha > 1$

① in both numerator & denominator, division by  $c^2$  is there

②

★ Numerator signs are same for both  $x$  &  $t$

Also note, inverse Lorentz transformation

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}$$

$$y = y'$$

$$z = z'$$

$$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

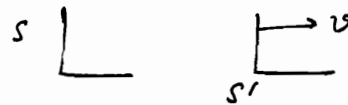
⊛ Note that the frames are equivalent

frame 2 going @  $v$  wrt frame 1

↳ frame 1 going @  $-v$  wrt frame 2.

⊛ Also note that sign of  $x$  and  $x'$  concomitant with the diagram

(3)



$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

⊛ Lorentz transforms reduce to Galilean Transforms

if  $\frac{v^2}{c^2} \ll 1$

i.e. even if  $v = 10^6$  m/s

$$\frac{v^2}{c^2} \approx 0$$

Lorentz invariants (Can be left for now.... continue reading further)

①  $x^2 + y^2 + z^2 - c^2t^2$

②  $dx^2 + dy^2 + dz^2 - c^2dt^2$

③  $dx dy dz dt$

④  $\left[ \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2} - \frac{\partial}{c^2 \partial t^2} \right]$

de - Alembertian

$$\square^2 = \nabla^2 - \frac{1}{c^2} \left( \frac{\partial^2}{\partial t^2} \right)$$

$$5) \quad \beta^2 = \frac{E^2}{c^2}$$

To Prove

$$x^2 + y^2 + z^2 - c^2 t^2 = x'^2 + y'^2 + z'^2 - c^2 t'^2$$

$$\text{R.H.S.} = x'^2 + y'^2 + z'^2 - c^2 t'^2$$

$$= \frac{(x - vt)^2}{1 - v^2/c^2} + y^2 + z^2 - c^2 \frac{\left(t - \frac{vx}{c^2}\right)^2}{1 - (v^2/c^2)}$$

$$= \frac{1}{\left(1 - \frac{v^2}{c^2}\right)} \left[ x^2 + v^2 t^2 - 2xvt - c^2 t^2 - \frac{c^2 v^2 x^2}{c^4} + \frac{2c^2 vx t}{c^2} \right]$$

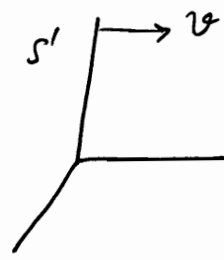
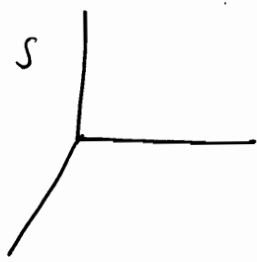
$$= \frac{1}{\left(1 - \frac{v^2}{c^2}\right)} \left[ x^2 \left(1 - \frac{v^2}{c^2}\right) + c^2 t^2 \left[1 - \frac{v^2}{c^2}\right] \right]$$

$$= \left[ x^2 - c^2 t^2 + y^2 + z^2 \right]$$

### SIMULTANEITY

2 events occurring at same : simultaneous events

2 ~~simultaneous~~ events may be simultaneous in 1 inertial frame and may not be simultaneous in other frame, & vice versa.



Let 2 events be simultaneous in frame S i.e.

$$t_1 = t_2$$

$$x_1 \neq x_2$$

$$t_1' = \frac{t_1 - \frac{vx_1}{c^2}}{\sqrt{1 - v^2/c^2}}$$

$$t_2' = \frac{t_2 - \frac{vx_2}{c^2}}{\sqrt{1 - v^2/c^2}}$$

$$t_1' - t_2' = \frac{\frac{v}{c^2} (x_2 - x_1)}{\sqrt{1 - \frac{v^2}{c^2}}} \neq 0$$

$$\Rightarrow t_1' \neq t_2'$$

Hence events are not simultaneous in S'

→ We can prove vice versa via Inverse Lorentz transforms

# Consequences of Simultaneity & Lorentz transforms

## (1) Length Contraction / LORENTZ FITZGERALD CONTRACTION At same time t

Accurate lengths are measured only when it is at rest. All the measurement in body's frames are accurate. All the proper measurement in  $S'$ .

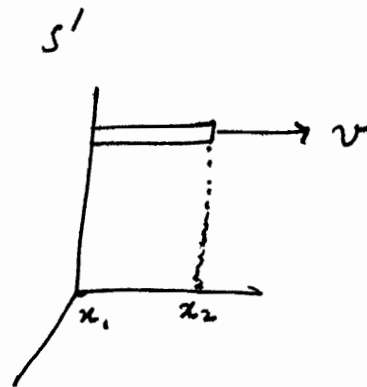
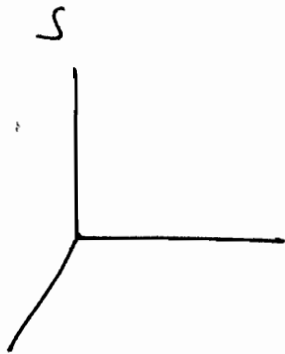
All the moving lengths are contracted by a factor of

$$\sqrt{1 - v^2/c^2} \quad \text{i.e.} \quad L = L_0 \sqrt{1 - \frac{v^2}{c^2}} \quad \text{along the}$$

direction of motion.

[Proper length, when body measured in its own frame.]

$L$  measured in  $S$



Correct

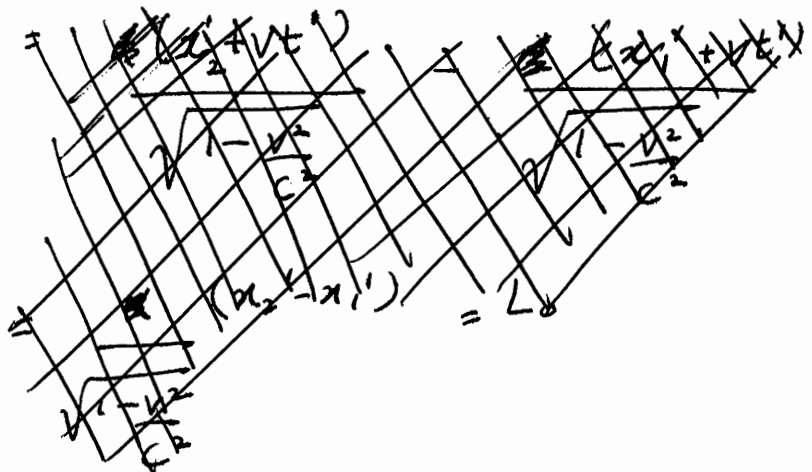
$$\text{length} = [x_2' - x_1'] = L_0$$

Body is measured at rest. We can measure anyhow

From frame  $S$ ,  $L = x_2 - x_1$

We have to take precaution of measuring  $x_2$  and  $x_1$  at time  $t$  simultaneously.

(\*)



$$x_1' = \alpha (x_1 - vt)$$

$$x_2' = \alpha (x_2 - vt)$$

$$\boxed{x_1' - x_2'} = \alpha \boxed{(x_1 - x_2)}$$

$\rightarrow L_0 = \frac{L}{\sqrt{1 - \frac{v^2}{c^2}}}$

$$\Rightarrow \underline{L = L_0 \sqrt{1 - \frac{v^2}{c^2}}}$$

$\rightarrow$  We cannot do the inverse. 2 positions measured simultaneously in  $S'$  may not be simultaneous for  $S$ .

But for 2 positions measured simultaneously in  $S$ , may or may not be simultaneous for  $S'$  which is immaterial b'coz for  $S'$  body is at rest.

- Note that length measured in  $S'$  : Proper length  $L_0$
- length measured in  $S$  keeping  $\Delta t = 0$  : Correct length  $L$

$$\boxed{d = \frac{L_0}{\alpha}}$$

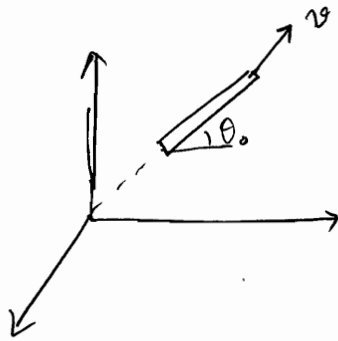


# MECHANICS (14)

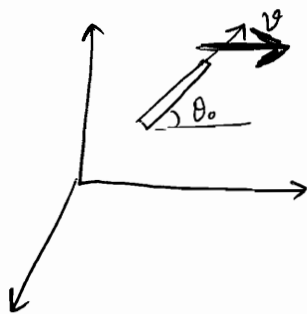
- Time dilation
- velocity addition
- Minkowsky

30/11/11

• Remember, length is contracted along the direction of motion.



$$L = L_0 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$$



$$L_y = L_0 \sin \theta_0$$

$$L_x = L_0 \cos \theta_0 \sqrt{1 - \frac{v^2}{c^2}}$$

$$L = \sqrt{L_y^2 + L_x^2}$$

$$\theta = \tan^{-1} \left[ \frac{\sin \theta_0}{\cos \theta_0 \sqrt{1 - \frac{v^2}{c^2}}} \right]$$

Note that in "time dilation", we talk of time interval  $(t_2 - t_1)$  between 2 events 1 and 2 and not just time  $t$ .

Time dilation

At same  $x'$

★ Clock is fixed in frame  $S'$  at a point say  $x'$ . It measures two times  $t_1'$  and  $t_2'$

All moving clocks go slow.

Time Interval  $T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}}$

i.e.  $T > T_0$

⊙ दोनों के लिए दूसरे का meter scale छोटा हो जाएगा!!

दोनों के लिए दूसरे का time slow हो जाएगा!!

i.e. even if I apply correction  
 $\Rightarrow \frac{\alpha dt'}{\alpha t} = \left(\frac{dt}{dt'}\right) = c$

$$dt^* = \frac{dt'_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

← general notation for moving frame time interval is  $T$

⊙  $t_1', t_2'$  should be measured at const.  $x'$  ★

$$t_1 = \alpha \left( t_1' + \frac{vx_1'}{c^2} \right)$$

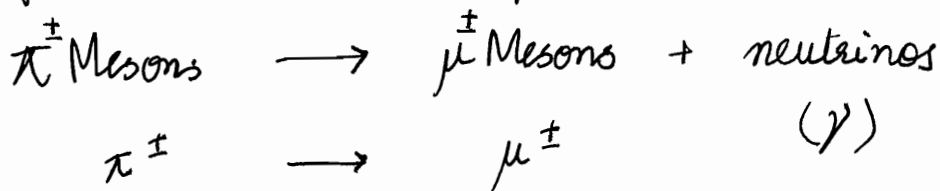
$$t_2 = \alpha \left( t_2' + \frac{vx_2'}{c^2} \right)$$

Note that inverse Lorentz transformation are reqd. to prove time dilation

$$\boxed{t_2 - t_1} = \alpha \boxed{[t_2' - t_1']} \Rightarrow \boxed{T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Do not think in terms of unit interval or time = k multiplied by unit interval etc.  $\text{अथवा}$

✓  $\pi$ -Mesons are unstable i.e. they have a lifetime during which they decay out. (refer end of lecture)



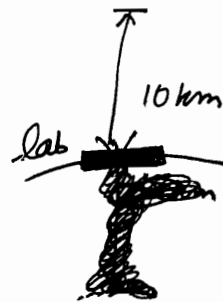
$\pi$  Mesons are present in cosmic rays near poles at 10 km height.

Avg lifetime :  $2.2 \times 10^{-6}$  seconds (in mesons frame)

Now we get  $\mu$  Mesons in laboratory 10 km below.

$$v_{\text{Meson}} = 0.998c$$

$$d = v \times T \approx 600 \text{ m}$$



So how we observe at a distance of 10 km??

$$T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}} \approx 17 \text{ times}$$

Hence the reason for observation.

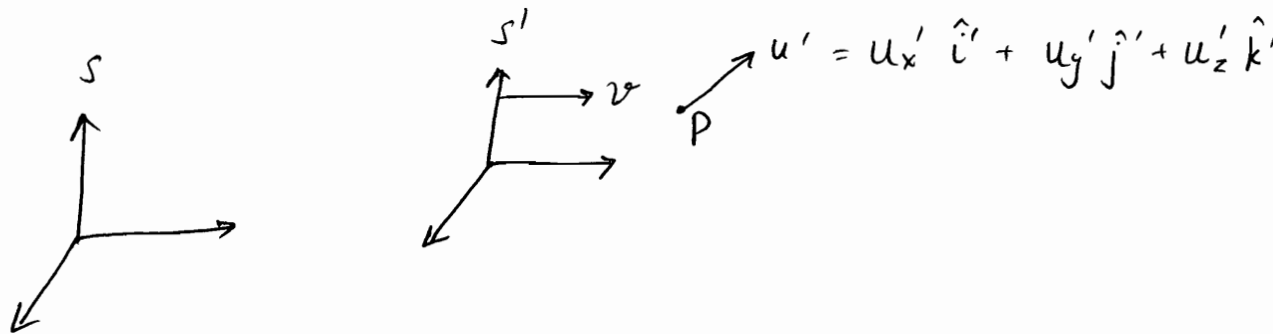
# Twin's Paradox

→ Proper Time: time measured in moving frame.

↳ twin on earth ages by 17 years  
twin on spacecraft ages by 1 year

other time is dilated time.

## Law of Composition of velocities in STR



$S'$  Observer moving with velocity  $v$  w.r.t  $S$   
Particle moving with velocity  $u'$  w.r.t  $S'$

$$v_{p, S'} = u'$$

$$u'_x = \frac{dx'}{dt'}$$

$$u'_y = \frac{dy'}{dt'}$$

$$u'_z = \frac{dz'}{dt'}$$

} known quantities

## Velocity Addition Theorem

$$\rightarrow u_x = \left[ \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} \right]$$

$$u_y = \left[ \frac{u'_y \sqrt{1 - v^2/c^2}}{1 + \frac{u'_x v}{c^2}} \right] \quad u_z = \left[ \frac{u'_z \sqrt{1 - v^2/c^2}}{1 + \frac{u'_x v}{c^2}} \right]$$

$$\text{If } v \ll c \Rightarrow u_x = u'_x + v$$

$$u_y = u'_y$$

$$u_z = u'_z$$

# derivation

$$x = \alpha (x' + vt')$$

$$\boxed{dx = \alpha (dx' + v dt')}$$

$$y = y'$$

$$dy = dy'$$

$$z = z'$$

$$dz = dz'$$

$$t = \alpha \left( t' + \frac{vx'}{c^2} \right)$$

$$\boxed{dt = \alpha \left( dt' + \frac{v}{c^2} dx' \right)}$$

$$u_x = \frac{dx}{dt} = \frac{dx' + v dt'}{dt' + \frac{v}{c^2} dx'}$$

$$= \frac{u_x' + v}{1 + \left( \frac{v}{c^2} \right) u_x'}$$

$$u_y = \frac{dy}{dt} = \frac{dy'}{\alpha \left( dt' + \frac{v}{c^2} dx' \right)}$$

$$= \frac{v_y'}{\alpha \left[ 1 + \frac{v}{c^2} u_x \right]} = \frac{v_y' \sqrt{1 - v^2/c^2}}{1 + \left( \frac{v}{c^2} \right) u_x}$$

$$u_z = \frac{v_z' \sqrt{1 - v^2/c^2}}{1 + \left( \frac{v}{c^2} \right) u_x}$$

Using Lorentz equations, we can do vice versa also.

$u_x, u_y, u_z$  known

$v_x$  known

$u_x', u_y', u_z': ??$

$$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$u_y' = \frac{u_y \sqrt{1 - v^2/c^2}}{1 - \left(\frac{u_x v}{c^2}\right)}$$

$$u_z' = \frac{u_z \sqrt{1 - v^2/c^2}}{1 - \left(\frac{u_x v}{c^2}\right)}$$

signs are similar to Lorentz Transformation

no need to remember them... can be derived in a jiffy!!

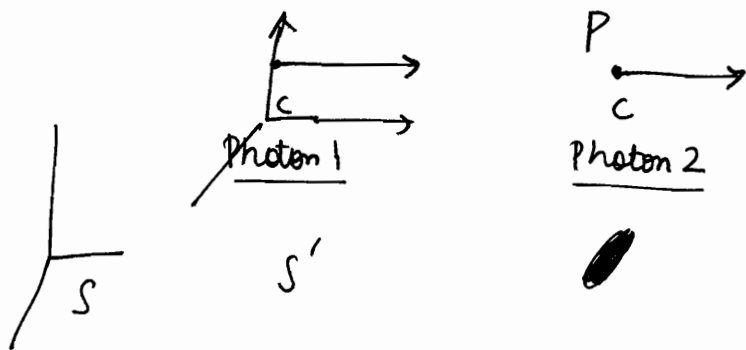
derivation

$$x' = \alpha (x - vt) \quad dx' = \alpha (dx - v dt)$$

$$dt' = \alpha \left( dt - \frac{v}{c^2} dx \right)$$

$$u_x' = \frac{dx'}{dt'} = \frac{\alpha \left( \frac{dx}{dt} - v \right)}{\alpha \left( 1 - \frac{v}{c^2} \frac{dx}{dt} \right)} = \left[ \frac{u_x - v}{1 - \frac{v}{c^2} u_x} \right]$$

$$u_y' = \frac{dy}{dt'} = \frac{\frac{dy}{dt}}{\alpha \left( 1 - \frac{v}{c^2} \frac{dx}{dt} \right)} = \frac{u_y}{\left( 1 - \frac{v}{c^2} u_x \right)}$$



$$u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}} = \frac{c + c}{1 + 1} = c$$

Take the limit for velocity of the frame

$$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \lim_{v \rightarrow c} \frac{c - v}{1 - \frac{v}{c}} = \lim_{v \rightarrow c} c = \underline{\underline{c}}$$

= + c

Lorentz Equations can be written in very compact form.

let

$$x_1' = x' = \alpha (x - vt) = \alpha (x_1 - vt)$$

$$x_2' = y' = y = x_2$$

$$x_3' = z' = z = x_3$$

~~the~~

$$t' = \alpha \left( t - \frac{v x}{c^2} \right)$$

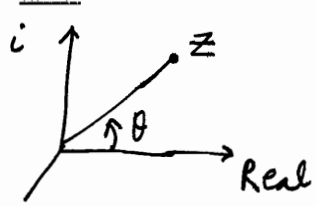
$$\alpha = \frac{1}{\sqrt{1 - v^2/c^2}}$$

let  $\beta = \left( \frac{v}{c} \right)$

$$x_4 = ict$$

Space-time continuum : Space and time are interrelated

$$e^{i\frac{\pi}{2}} = i$$



$$e^{i\theta} = \cos\theta + i\sin\theta = z$$

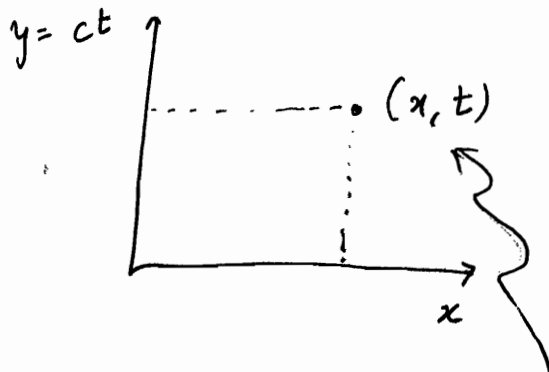
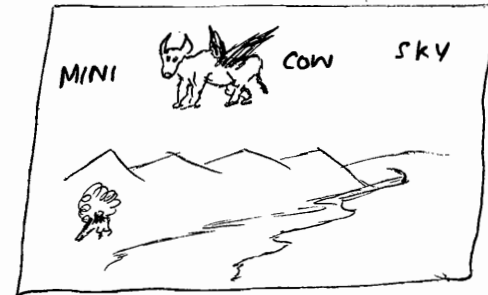
$$\vec{z} = r\cos\theta + r\sin\theta i$$

$$= r e^{i\theta}$$

Note that  $y, z$  are not changing. Only  $x$  and  $t$  are changing.

### Minkowsky's 4 dimensional space

In space-time continuum,



$$x_1 = x$$

$$x_2 = y$$

$$x_3 = z$$

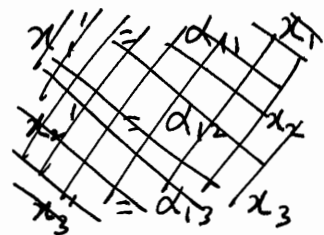
$$x_4 = ict$$

Now event will be represented as  $(x, t)$

$$x'_\mu = \sum_\nu \alpha_{\mu\nu} x_\nu$$

~~$x'_\mu = \sum_\nu \alpha_{\mu\nu} x_\nu$~~

etc



$$x_1' = \alpha \left( x_1 - \frac{vct}{c} \right) = \alpha \left[ x_1 + \frac{i^2 vct}{c} \right]$$

$$\Rightarrow x_1' = \alpha x_1 + 0 x_2 + 0 x_3 + i \alpha \beta x_4$$

$$x_2' = 0 x_1 + 1 x_2 + 0 x_3 + 0 x_4$$

$$x_3' = 0 x_1 + 0 x_2 + 1 x_3 + 0 x_4$$

$$t' = \alpha \left[ t - \frac{vx}{c^2} \right]$$

$$\Rightarrow ict' = \alpha ict - \alpha \frac{v}{c^2} i \alpha x$$

$$\Rightarrow x_4' = -i \alpha \beta x_1 + 0 x_2 + 0 x_3 + \alpha x_4$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 & i \alpha \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i \alpha \beta & 0 & 0 & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$x_\mu' = \alpha_{\mu\nu} x_\nu$$

→ Antisymmetric about the diagonal



## Verification of Time dilation (Remember the following parameter)

For  $\mu$  meson produced in atmosphere,  $v < c$  and  
mean lifetime in a frame that is at rest =  $2.2 \times 10^{-6} \text{ s}$

$$\Rightarrow d < 2.2 \times 10^6 \text{ c} = \underline{0.66 \text{ km}}$$

(expected distance travelled by them)

$\therefore$  in absence of any dilation effect,  $\mu$ -meson have no possibility of being detected at earth's surface, while being produced at top of atmosphere. But experimentally they have been observed.

This becomes possible only when we consider the lifetime of a fast  $\mu$ -meson to be much longer than the measured lifetime of a  $\mu$ -meson at rest. The  $\mu$ -meson of cosmic rays, detected on the Earth's surface have an energy of the order corresponding to  $v \approx 0.999 \text{ c}$  or  $\gamma = 30$

$$\Rightarrow \Delta t \text{ as observed from Earth} = \underline{30 \Delta T} = 30 \times 2.2 \times 10^{-6} \\ = 6.6 \times 10^{-5} \text{ s}$$

$$\Rightarrow \text{distance travelled} \approx 20 \text{ km.}$$

PROOF OF  
RELATIVITY

⊛ Note that  $x^2 + y^2 + z^2 = c^2 t^2$  only if ~~the~~ dealing with motion of light.

⊛ Note that even while deriving Lorentz transformations, we used

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = x^2 + y^2 + z^2 - c^2 t^2$$

We did not equate them to zero. Hence, very general derivation.

⊛ To prove  $|p_{\mu}|$  is Lorentz invariant, write down Lorentz transformations and prove via L.H.S  $\rightarrow$  R.H.S

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = x^2 + y^2 + z^2 - c^2 t^2$$

# MECHANICS (15)

- Covariance
- space-time interval 01/12/11
- momentum four vector

$$x'_\mu = \alpha_{\mu\nu} x_\nu$$

$$\alpha_{1\nu} : 1^{\text{st}} \text{ row}$$

$$x'_1 = \alpha_{1\nu} x_\nu = \sum_{\nu=1}^4 \alpha_{1\nu} x_\nu$$

Note that importance of  $\alpha_{\mu\nu}$  tensor

If movement along y direction

$$\text{Lorentz Transform } \alpha_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & i\alpha\beta \\ 0 & 0 & 1 & 0 \\ \alpha & -i\alpha\beta & 0 & \alpha \end{bmatrix}$$

law of transformation  $x'_\mu = \alpha_{\mu\nu} x_\nu$  remains same  
just tensor is changed a little bit.

If movement along z axis

$$\text{Lorentz Transform } \alpha_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & i\alpha\beta \\ 0 & 0 & -i\alpha\beta & \alpha \end{bmatrix}$$

⊙ Note that  
Minkowski's matrix  
of transformation is  
ANTI SYMMETRIC about  
diagonal  $\begin{bmatrix} \oplus \\ \ominus \end{bmatrix}$

Note that  $\alpha_{\mu\nu}$  is constant

Note the representations  
 $x_\mu, \alpha_{\mu\nu}$

$$\boxed{x'_\mu = \alpha_{\mu\nu} x_\nu}$$

Lorentz Transformation  
in 4-vector form

$$\Rightarrow \dot{x}'_\mu = \alpha_{\mu\nu} \dot{x}_\nu$$

If 4-vector of any vector  $A$  is known <sup>(in  $S$ )</sup> it can be converted to  $S'$  using the above Lorentz transform.

Now is it that simple? No; only that vector  $A_\mu$  can be represented as 4-vector whose mod i.e.  $|\vec{A}_\mu|$  is 4-vector Lorentz invariant.

$$x_\mu = (\underbrace{\vec{r}, ct}) \text{ in frame } S$$

Representation  
of 4 vector

✓ In our syllabus, we have

- Position Four-Vector
- Velocity Four-Vector
- Momentum Four-Vector

if  $x_\mu \cdot x_\mu$  is Lorentz invariant.

$$\text{i.e. } x_\mu \cdot x_\mu = r^2 - c^2 t^2$$

$$\Rightarrow x_\mu \cdot x_\mu = x^2 + y^2 + z^2 - c^2 t^2 \text{ which is Lorentz invariant.}$$

Covariance (refer end of lecture)

Covariance = invariance

If any physical law or identity retains in form while going from 1 inertial frame to other inertial frame, we say "invariance w.r.t coordinate transform" or "Covariance of law".

<give example>

If  $x_\mu$  is a 4-vector

$\Rightarrow |x_\mu|^2$  is Lorentz invariant

We know,

$\cdot dx'_\mu = \alpha_{\mu\nu} dx_\nu$

This analysis of space-time interval invariance is important

You can replace  $dx, dt$  by

$\Delta x$  or  $(x_2 - x_1)$   
 $\Delta t$  or  $(t_2 - t_1)$   
 for better understanding

- (1)

$$\begin{cases} dx'_1 = \alpha_{11} dx_1 + \alpha_{12} dx_2 + \alpha_{13} dx_3 + \alpha_{14} dx_4 \\ \quad = \alpha dx_1 + i\alpha\beta dx_4 \\ dx'_2 = dx_2 \\ dx'_3 = dx_3 \\ dx'_4 = -i\alpha\beta dx_1 + \alpha dx_4 \end{cases}$$

~~$\Rightarrow dx'_1 = \alpha dx_1 + i\alpha\beta dx_4 - i\alpha\beta dx_1 + \alpha dx_4$~~   
 ~~$= (\alpha - i\alpha\beta) dx_1 + (i\alpha\beta + \alpha) dx_4$~~

$\Rightarrow \left. \begin{aligned} dx'_1 &= \alpha (dx - v dt) \\ dt' &= \alpha \left( dt - \frac{v}{c^2} dx \right) \end{aligned} \right\} \text{this we already know}$

Space time interval  $s_{12}$  is defined as:  $s_{12}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2$   
 similarly  $s'_{12} = (x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2 - c^2(t'_2 - t'_1)^2$

If  $dx_\mu$  is a 4 vector, square of its norm must be Lorentz invariant.

ie.  $dx'_\mu{}^2 = dx_1'^2 + dx_2'^2 + dx_3'^2 + dx_4'^2$   
 $= \alpha^2 [dx_1^2 + \beta^2 dx_4^2 + 2i\beta dx_1 dx_4 + \beta^2 dx_1^2 + dx_4^2 - 2i\beta dx_1 dx_4] + dx_2^2 + dx_3^2$   
 $= \alpha^2(1 - \beta^2) (dx_1^2 + dx_4^2) + dx_2^2 + dx_3^2$

$$dx_\mu'^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 = dx_\mu^2$$

\* Therefore, the time interval as well as space interval between two events is not the same in two inertial frames. Therefore unlike in Newtonian mechanics or Galilean Transformations, in STR time and space intervals are not invariant separately.

$$dx_\mu^2 = ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - c^2 dt^2$$

↑  
[ ds is the length element  
in 4 vector space ]

However, we see that space time interval  $s_{12}$  in the theory of relativity is independent of the frame of reference & absolutely specifies the separation between 2 events in Minkowski world.

o2 Events are separated in space or in time or both.

Space like interval:  $ds^2 > 0$

i.e.  $dx_1^2 + dx_2^2 + dx_3^2 > c^2 dt^2$

Time like interval:  $ds^2 < 0$

i.e.  $dx_1^2 + dx_2^2 + dx_3^2 < c^2 dt^2$

Light cone:  $ds^2 = 0$

i.e.  $dx_1^2 + dx_2^2 + dx_3^2 = c^2 dt^2$

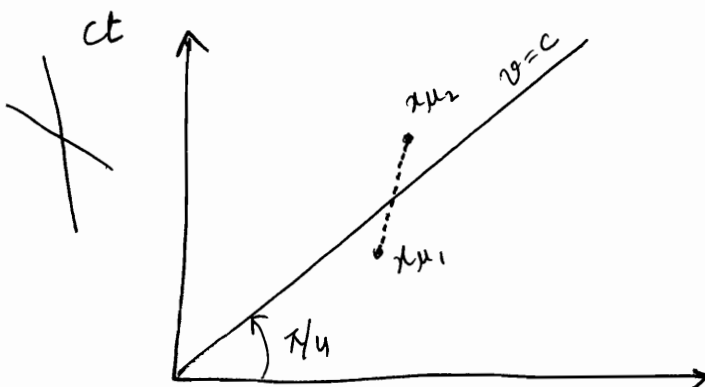
i.e.  $\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2 = c^2$

i.e.  $v^2 = c^2$  or  $v = c$  ✓

\* Note that interval  $s_{12}$  between 2 events: nothing is observable & it's a mathematical entity. In theory of relativity, time & space lose their mutually independent significance & in fact they occur as constituents of the more fundamental entity, called SPACE-TIME.

3 dimensions of space & 1 dimension of time merge together to form FOUR DIMENSIONAL SPACE TIME CONTINUUM.

(refer end of lecture)



$$ds^2 = (x_{\mu_2} - x_{\mu_1})^2 = dx_\mu^2$$

interval or world line

$$dx^2 - c^2 dt^2 = dx'^2 - c^2 dt'^2$$

Can be left

Proof of 2 Phenomenon using above expression!!  
Length Contraction

$$L^2 = L_0^2 \left[ 1 - \frac{c^2 v^2}{c^4} \right] = \frac{L_0^2}{\alpha^2}$$

$$dx^2 = dx'^2 \left[ 1 - c^2 \left( \frac{dt'}{dx'} \right)^2 \right]$$

$$[dt=0] \Rightarrow \boxed{L = \frac{L_0}{\alpha}}$$

at  $dt=0$

**INVERSE**

$$t = \alpha \left[ t' + \frac{v}{c^2} x' \right]$$

$$dt = \alpha \left[ dt' + \frac{v}{c^2} dx' \right]$$

$$\Rightarrow \frac{dt'}{dx'} = \left( \frac{-v}{c^2} \right)$$

Time Dilation

at  $dx'=0$  we measure time

$$dx^2 - c^2 dt^2 = -c^2 dt'^2$$

$$-dt' = \frac{dx^2 - c^2 dt^2}{c^2} = dt^2 \left[ \frac{v^2}{c^2} - 1 \right]$$

$$dt' = dt \sqrt{1 - v^2/c^2}$$

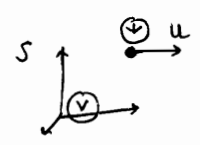
$$\Rightarrow dt = \frac{dt'}{\sqrt{1 - v^2/c^2}} \Rightarrow T = \frac{T_0}{\sqrt{1 - v^2/c^2}}$$

$$T = \int \frac{1}{\alpha} dt$$

$$= \int \sqrt{1 - \frac{v^2}{c^2}} dt$$

$$= \int \sqrt{1 - \frac{1}{c^2} \left[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \right]} dt$$

Is velocity 4-vector?



$dt'$  is defined as the time measured on a clock placed on a moving body

$$u_\mu = \left( \frac{dx_\mu}{dt} = \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, ic \frac{dt}{dt} \right) \quad \left( \frac{d\tau}{dt} \right) = \left( \frac{1}{\alpha} \right)$$

Here  $\alpha = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$

$|u_\mu|^2 = v^2 - c^2$  : not Lorentz invariant which we use in Minkowski as this  $\alpha$  has nothing to do with any other frame.

NO

In 4-vectors time derivatives are taken w.r.t. Proper time  $\tau = t'$ . As 'proper time' is invariant,

this guarantees that Proper-Time-Derivative of any four-vector is itself a four-vector. Also  $\frac{d\tau}{dt} = \left( \frac{1}{\alpha} \right)$

# Four-Velocity

def  $u_\mu = \frac{dx_\mu}{dt} = \left( \frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}, \frac{dx_4}{dt} \right)$  → Note the 4th component of four-velocity...

$= \left( \alpha \frac{dx_1}{dt}, \alpha \frac{dx_2}{dt}, \alpha \frac{dx_3}{dt}, \alpha \frac{dx_4}{dt} \right)$

$u_\mu^2 = \alpha^2 [u^2 - c^2]$  &  $\alpha = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$

$dt = \alpha dt'$

⊙  $\alpha$  is consistent & just  $\alpha$  here is used in minihowsky and not the one used in Lorentz invariant!

→ Velocity Four-Vector is a hybrid & therefore not used for transformations

$u_\mu = \alpha \left( \frac{dx_\mu}{dt} \right) = \underline{\underline{(\alpha \vec{u}, i\alpha c)}}$

Four-Velocity =  $\begin{bmatrix} \alpha u_x \\ \alpha u_y \\ \alpha u_z \\ i\alpha c \end{bmatrix}$

# Four Momentum

def  $p_\mu = m_0 u_\mu = \alpha m_0 \left( \frac{dx_\mu}{dt} \right)$  [4-vector representation]

Four momentum =  $m_0$  (Four Velocity)

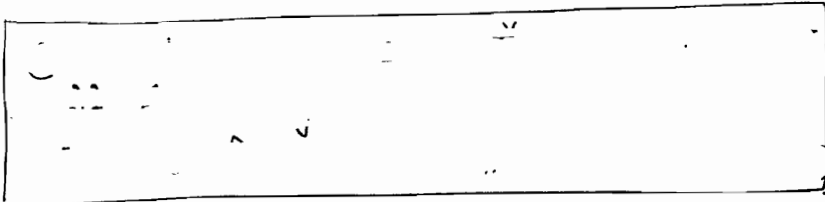
=  $\begin{bmatrix} \alpha m_0 u_x \\ \alpha m_0 u_y \\ \alpha m_0 u_z \\ i\alpha m_0 c \end{bmatrix}$

=  $\begin{bmatrix} m u_x \\ m u_y \\ m u_z \\ i m c \end{bmatrix}$

=  $\begin{bmatrix} p_x \\ p_y \\ p_z \\ i E/c \end{bmatrix}$

✓ velocity 4 vector :  $\alpha \left( \frac{d\vec{r}}{dt}, ic \right)$

✓ momentum 4 vector :  $p_\mu = m_0 u_\mu = \alpha m_0 \frac{dx_\mu}{dt}$



$m \left( \frac{dx_\mu}{dt} \right)$

⊙ Note that relativistic momentum is defined as  $p = \frac{1}{\sqrt{1 - v^2/c^2}} \cdot m_0 \cdot v$  where  $v$  is the velocity of the particle

$p = \frac{1}{\sqrt{1 - v^2/c^2}} \cdot m_0 \cdot v$

where  $v$  is the velocity of the particle



$$p_\mu = (dm_0 v_x, dm_0 v_y, dm_0 v_z, dm_0 i c)$$

$$= (dm_0 v_x, dm_0 v_y, dm_0 v_z, i \left[ \frac{E}{c} \right]) \quad \text{Using } E = mc^2$$

$$\left[ p^2 + m_0^2 c^2 = \frac{E^2}{c^2} \right]$$

Note that we can write  $p_\mu$  as 4-vector since  $d^2[(m_0 v_x)^2 + (m_0 v_y)^2 + (m_0 v_z)^2 + m_0^2 c^2] = m_0^2 d^2[v^2 - c^2] = -\underline{m_0^2 c^2} = \text{const.}$

$$p^2 - \frac{E^2}{c^2} = -m_0^2 c^2 = \text{const.} \Rightarrow \text{invariant.}$$

$$p_\mu = m_0 \frac{dx_\mu}{dt} = \left( \vec{p}, \frac{iE}{c} \right)$$

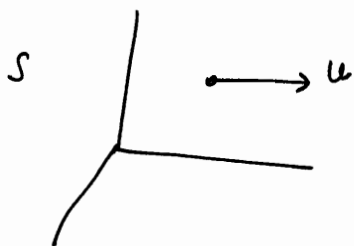
$$\vec{p} = m\vec{v} = dm_0 \vec{v}$$

$$E = mc^2 = dm_0 c^2$$

$$(mc^2)^2 = (\text{Energy})^2 = (pc)^2 + (m_0 c^2)^2$$

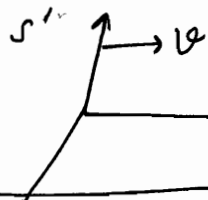
↑  
relativistic momentum

AIM



$$p_\mu \text{ in } S = \left( p, \frac{iE}{c} \right)$$

$$p'_\mu \text{ in } S' = ?$$



Four-Momentum

$$P_\mu = \left( p_x, p_y, p_z, \frac{iE}{c} \right)$$

$$\underline{P_\mu = \left( \vec{p}, \frac{iE}{c} \right)}$$

$$p' = (m' u_x', m' u_y', m' u_z')$$

$$E' = (m' c^2)$$

$$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$u_y' = \frac{u_y \sqrt{1 - v^2/c^2}}{1 - \frac{u_x v}{c^2}}$$

$$u'^2 = \sqrt{u_x'^2 + u_y'^2 + u_z'^2}$$

$$u_z' = \frac{u_z \sqrt{1 - v^2/c^2}}{1 - \frac{u_x v}{c^2}}$$

$$m' = \frac{m_0}{\sqrt{1 - \frac{u'^2}{c^2}}}$$

$$E' = \frac{m_0}{\sqrt{1 - \frac{u'^2}{c^2}}} c^2$$

if  $u_y = u_z = 0$

$$\Rightarrow E' = \frac{m_0 c^2}{\sqrt{1 - \frac{(u_x - v)^2}{c^2 (1 - \frac{u_x v}{c^2})^2}}}$$

$$= m_0 \left[ 1 - \frac{u_x v}{c^2} \right] c^2 / \sqrt{\left( 1 - \frac{u_x^2}{c^2} \right) \left( 1 - \frac{v^2}{c^2} \right)}$$

$$\Rightarrow E' = \frac{m_0 \left(1 - \frac{u_x v}{c^2}\right) c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$= \frac{m c^2 \left(1 - \frac{u_x v}{c^2}\right)}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

moving mass in frame S

$$= \frac{m c^2 - m u_x v}{\sqrt{1 - v^2/c^2}}$$

$$m = \frac{m_0}{\sqrt{1 - \frac{u_x^2}{c^2}}}$$

$$m = \frac{m_0 \left(1 - \frac{u_x v}{c^2}\right)}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

⊛ Note that mass transformation also takes place ...  
via  $E' = m'c^2$   
 $E = mc^2$

$$E' = \frac{E - v p_x}{\sqrt{1 - v^2/c^2}}$$

$p$  along the direction of movement of  $S'$

Energy Transformation

Alternatively

$$p_4' = \alpha_{41} p_1 + \alpha_{44} p_4 = -i\alpha\beta p_1 + \alpha p_4$$

$$\frac{iE'}{c} = \alpha \left[ \frac{iE}{c} - \frac{iv}{c} p_x \right]$$

$$E' = \frac{E - v p_x}{\sqrt{1 - (v^2/c^2)}} \quad \text{Energy Txn.}$$

$$p_1' = \alpha_{11} p_1 + \alpha_{14} p_4 = \alpha p_1 + i\alpha\beta p_4$$

$$p_x' = \alpha \left[ p_x + i\beta \cdot \frac{iE}{c} \right]$$

$$p_x' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left[ p_x - \frac{vE}{c^2} \right]$$

Momentum Txn.

$$p' = (p_x', p_y', p_z')$$

$$= (m' u_x', m' u_y', m' u_z')$$

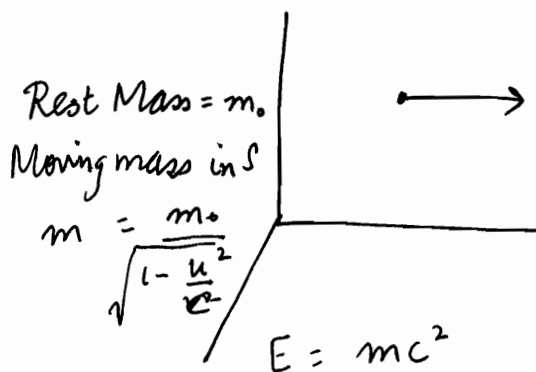
$$= \left[ \frac{m \left(1 - \frac{u_x v}{c^2}\right)}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{(u_x - v)}{\left(1 - \frac{u_x v}{c^2}\right)}, 0, 0 \right]$$

$$= \left[ \frac{m u_x - m v}{\sqrt{1 - \frac{v^2}{c^2}}}, 0, 0 \right]$$

$$p' = \left[ \frac{p_x - \frac{E v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \right]$$

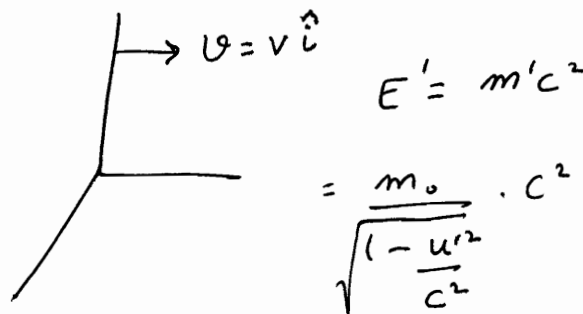
Momentum Transformation

### Summary



$$p_\mu = \left( \vec{p}, \frac{iE}{c} \right)$$

$$\vec{p} = (p_x, 0, 0)$$



$$u' = \left[ \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \right]$$

Momentum Energy Transforms  $E' = \frac{E - v p_x}{\sqrt{1 - \frac{v^2}{c^2}}}$

$$p_x' = \frac{p_x - E v / c^2}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad p_y' = p_y, \quad p_z' = p_z$$

→ All these can be avoided using Lorentz transformation of the 4 vector

$$p'_\mu = \alpha_{\mu\nu} p_\nu$$

Any vector is 4-vector if  $|\vec{A}|^2$  is Lorentz invariant.

### COVARIANCE

Covariance is the invariance of the "form" of physical laws under coordinate transformations. In particular, if any physical law or identity retains its form while going from one inertial frame to another inertial frame, we say that the "law is covariant" or "invariant w.r.t. coordinate transformation".

eg.  $x^2 + y^2 + z^2 - c^2 t^2 = 0$  in all frames

$p^2 - \frac{E^2}{c^2} = \text{const.}$  in all frames.

The essential idea is that coordinates do not exist a priori in nature, but are only tools used to describe the natural laws. Hence, they should not play any role in the formulation of fundamental physical laws.

The principle of covariance was proposed by Albert Einstein for STR, although, that theory was limited to space-time coordinate system, related to inertial frames.

Covariant Transformation is a rule that specifies how certain entities change under change of coordinates. Usually used for vectors & tensors, it describes new coordinates as a linear combination of old coordinates.

## WORLD REGIONS AND LIGHT CONE

space time interval,  $s_{12}$  between 2 events  $E_1(x_1, y_1, z_1, t_1)$  and  $E_2(x_2, y_2, z_2, t_2)$  is given by

$$s_{12}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2 (t_2 - t_1)^2$$

which is Lorentz invariant.

To study world region, we consider one event as  $O(0,0,0,0)$  and other  $E(x, y, z, ct)$

$$\Rightarrow s^2 = x^2 + y^2 + z^2 - c^2 t^2$$

Region A :  $s^2 > 0$

Region B :  $s^2 < 0$

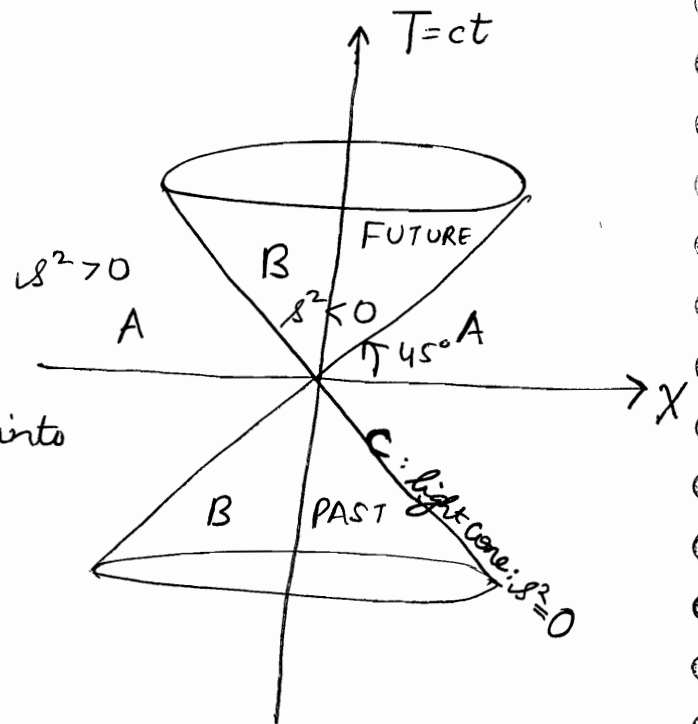
Sheet C :  $s^2 = 0$

Sheet C divides Minkowski World into regions A & B and is given by

$$x^2 + y^2 + z^2 = c^2 t^2$$

i.e.  $r = ct$

It's called Null Cone (as  $s^2 = 0$ ) or light cone.



Since dependence of interval is on  $t^2$ , it's satisfied by both  $+t$  (future) and  $-t$  (past) values.

### Twins Paradox

Consider twin A and B, each 20 years of age. Twin A remains at rest at the origin  $O$  while twin B takes a round trip space voyage to a star with  $v = 0.99c$  relative to A. Star is 10 light years away from  $O$ .

When B finishes his journey & returns to  $O$ :

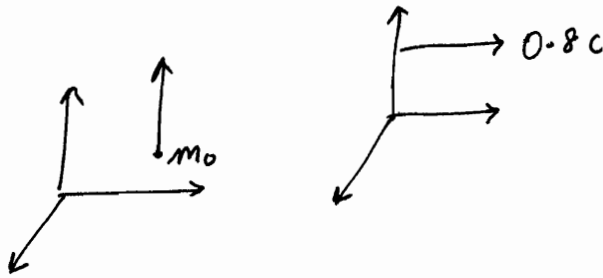
**Age of A** time taken by B for round trip =  $\frac{10 \times 2 \text{ light years}}{0.99c}$   
 $\Rightarrow$  According to A, age of A = 40.2 = 20.20 years

**Age of B** time taken by B, according to B =  $20.2 * \sqrt{1 - \frac{v^2}{c^2}} = \underline{\underline{22.8}}$   
 $\Rightarrow$  According to B, age of B = 22.8 years

# MECHANICS (15)

- Mass Energy equivalence 2/12/11
- Relativistic Force

Q33 / Tut 6

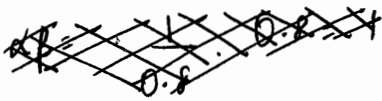


$$p_\mu = \left( \vec{p}, \frac{iE}{c} \right) \Leftrightarrow (m\vec{u}, imc)$$

$$= \left( \alpha m_0 (\vec{u}, ic) \right)$$

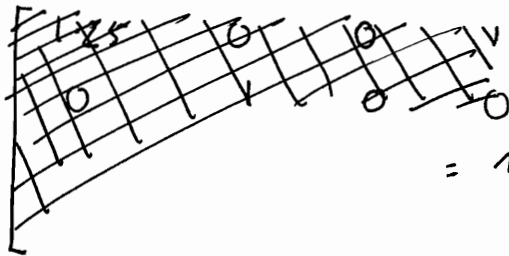
$$\left( \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \right)$$

$$= \frac{m_0}{0.8} [0, 0.6c, 0, ic]$$



$$= 1.25 [0, 0.6c, 0, ic]$$

$$= m_0 [0, 0.750c, 0, 1.25ic]$$



$$= m_0 c \left[ 0, \frac{3}{4}, 0, \frac{5}{4}i \right]$$

$$\alpha = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{0.6} \quad ; \quad \beta = \frac{v}{c} = 0.8$$

$$\odot dx' dy' dz' \neq dx dy dz$$

$$x' = \alpha(x - vt)$$

$$dx' = \alpha(dx - v dt)$$

$$dy' = dy$$

$$dz' = dz$$

$$t' = \alpha\left(t - \frac{v}{c^2}x\right)$$

$$dt' = \left[ \alpha dt - \frac{v}{c^2} dx \right]$$

$$\odot dx' dy' dz' dt' = dx dy dz dt$$

To prove  $dx' dt' = dx dt$

$$dx = dx' \sqrt{1 - \frac{v^2}{c^2}}$$

$$dt = \frac{dt'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Rightarrow dx \cdot dt = dx' \cdot dt'$$

$$dx' = \alpha(dx) \quad \left[ \text{when } dt = 0, \text{ Proper length} \right]$$

Now having inverse transform,

$$t = \alpha\left[t' + \frac{vx'}{c^2}\right]$$

$$dt = \alpha\left[dt' + \frac{v dx'}{c^2}\right] = \alpha(dt')$$

$\left[ \text{when } dx' = 0, \text{ Proper } \right]$   
~~time~~  
 time



to prove this simply write time dilation & length contraction relationships, or derive these relationships and use them directly over here.

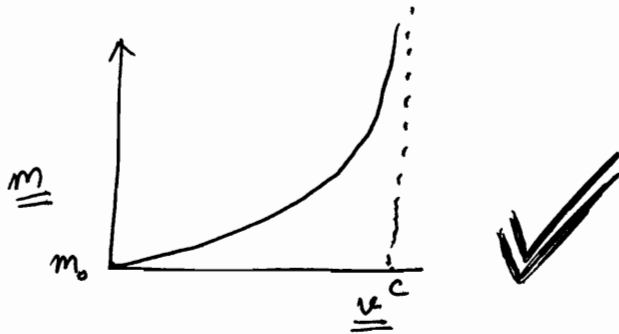


## Mass variation

$$m_0 \quad \quad \quad m \xrightarrow{v}$$

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \quad \leftarrow \text{formula valid only when } v < c$$

~~Always remember, c is constant and is 300,000,000 m/s.~~



No Proof required....

$$\frac{dm}{dv} = \frac{-1}{2} m_0 \left(1 - \frac{v^2}{c^2}\right)^{-3/2} \cdot \frac{-2v}{c^2}$$

$$dm = \frac{m_0 v dv}{c^2 \left(1 - \frac{v^2}{c^2}\right)^{-3/2}} = \frac{m v dv}{c^2 \left(1 - \frac{v^2}{c^2}\right)}$$

$$\boxed{dm = \frac{m v dv}{(c^2 - v^2)}}$$

# Mass - Energy Equivalence

If mass is known  $\Rightarrow$  Energy =  $mc^2$

If Energy is known  $\Rightarrow$  Mass Equivalent =  $\left(\frac{E}{c^2}\right)$

Note that  $m$  is moving mass

$$E = \frac{m_0 \cdot c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m_0 c^2$$

$$v=0 \Rightarrow \frac{E_0 = m_0 c^2}{\uparrow}$$

[rest mass energy]  
characteristic of body

Now  $v > 0 \Rightarrow E > E_0$  [more energy due to motion]

$$\Rightarrow \boxed{E = E_0 + T} \left[ \begin{array}{l} \text{Relativistic} \\ \text{kinetic energy is the energy} \\ \text{due to motion} \end{array} \right]$$

⊛ [ If the given energy is more than rest mass energy  $\Rightarrow$  it is relativistic case ] ✓

Relativistic kinetic Energy  $\boxed{T = (m - m_0) c^2}$  always valid!!!!

$$= \left[ m_0 \left[ 1 - \frac{v^2}{c^2} \right]^{-1/2} - m_0 \right] c^2 = m_0 \left[ 1 + \frac{v^2}{2c^2} - 1 \right] c^2$$

⊙ Neglecting  $(v^4/c^4)$  and above

$$= \boxed{\frac{1}{2} m_0 v^2}$$

$$1 \text{ A.M.U.} = \frac{1}{12} \text{ times } {}_6^{12}\text{C atom}$$

$$\approx 1.66 \times 10^{-27} \text{ kg}$$

$$\text{Energy Equivalent} = 1.66 \times 10^{-27} \times 9 \times 10^{16} \text{ Joules}$$

$$1 \text{ a.m.u @ rest} = \frac{1.66 \times 10^{-27} \times 9 \times 10^{16}}{1.6 \times 10^{-19}} \text{ eV}$$

$$1 \text{ a.m.u} = \underline{931 \text{ MeV}}$$

$$\text{Energy Equivalent of electron @ rest} = \frac{9.31 \times 10^{-31} \times 9 \times 10^{16}}{1.6 \times 10^{-19}}$$

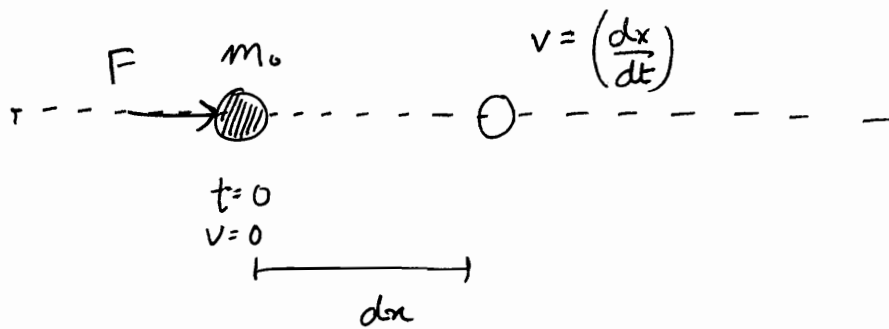
$$1 e^{-} \text{ mass} = 0.51 \text{ MeV}$$

$$\left[ \begin{array}{l} \text{if given } E_{\text{electron}} = 1 \text{ keV} \Rightarrow \frac{1}{2} m_e v^2 = 1 \text{ keV} \\ \text{if given } E_{\text{electron}} = 1 \text{ MeV} \Rightarrow mc^2 = 1 \text{ MeV} \end{array} \right]$$

$$1 \text{ proton mass} = 938 \text{ MeV}$$

$$1 \text{ neutron mass} = 940 \text{ MeV}$$

[ Untill and unless, written 'Relativistic kinetic Energy' ]  
otherwise  $E = mc^2$ . If given  $\uparrow$   $E = (m - m_0) c^2$  ]



$$dW = \Delta k \cdot E = dT$$

$$dW = F dx = \frac{d}{dt}(mv) dx$$

$$= \left[ m \left( \frac{dv}{dt} \right) + v \left( \frac{dm}{dt} \right) \right] dx$$

$$\Rightarrow dT = m \cdot v dv + v^2 dm$$

We know,  $m = m_0 \left[ 1 - \frac{v^2}{c^2} \right]^{-1/2}$  ✓

$$dm = \frac{mv}{c^2 - v^2} dv$$

✓: Postulate

$$\Rightarrow dT = m \cdot dm (c^2 - v^2) + v^2 dm$$

$$\left\{ \int_0^{KE} dT = c^2 \int_{m_0}^m dm \right\}$$

as k.E. energy increases from 0 to T, mass increases from  $m_0$  to  $m$ .

$$\Rightarrow \boxed{T = (m - m_0) c^2}$$

Total Energy = Rest Mass Energy + kinetic Energy

$$\Rightarrow \boxed{E} = \cancel{m_0 c^2} + (m - m_0) c^2 = \boxed{m c^2}$$

★ हम दो चीजों का  
Proof नहीं करते !!

①  $E_0 = m_0 c^2$   
②  $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$

$$\vec{p} = m\vec{v}$$

$v$ : velocity of particle

$$pc = mvc$$

$$\begin{aligned} \Rightarrow p^2 c^2 &= m^2 v^2 c^2 \\ &= \frac{m_0^2}{1 - \frac{v^2}{c^2}} v^2 c^2 \\ &= \frac{m_0^2 v^2 c^4}{c^2 - v^2} \end{aligned}$$

$$\begin{aligned} \checkmark \text{ RHS} &= p^2 c^2 + m_0^2 c^4 \\ &= m^2 v^2 c^2 + m^2 \left(1 - \frac{v^2}{c^2}\right) c^4 \\ &= m^2 v^2 c^2 + m^2 c^4 - m^2 v^2 c^2 \\ &= m^2 c^4 \\ &= E^2 = \text{LHS} \end{aligned}$$

$$p^2 c^2 + m_0^2 c^4$$

$$= \frac{m_0^2 v^2 c^4}{c^2 - v^2} + m_0^2 c^4$$

$$= \frac{m_0^2 c^6}{c^2 - v^2} = \frac{m_0^2}{1 - \frac{v^2}{c^2}} c^4 = m^2 c^4 = (mc^2)^2$$

$$\Rightarrow E^2 = p^2 c^2 + m_0^2 c^4$$

Contribution of Motion
Contribution of Rest Mass

$$\Rightarrow E^2 = (pc)^2 + (m_0 c^2)^2$$

Also

$$E = T + m_0 c^2$$

$$E^2 - p^2 c^2 = m_0^2 c^4$$

Also  $E'^2 - p'^2 c^2 = m_0^2 c^4$

$$\Rightarrow E^2 - p^2 c^2 : \text{Lorentz invariant}$$

$\Rightarrow$  express as Four-Momentum

dividing by  $-c^2$

$$p_{\mu}^2 = p^2 - \frac{E^2}{c^2} = -m_0^2 c^2 = p_{\mu}'^2$$

### Zero Rest Mass Particle

If  $m_0 = 0$

$$\Rightarrow E = pc = T$$

$$\propto p = [E/c]$$

$$= \frac{mc^2}{c} = mc$$

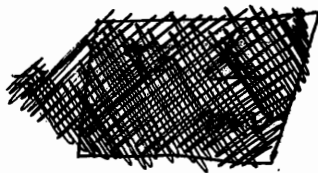
$$\Rightarrow \boxed{v = c}$$

Only possible if mass = 0

Its a wave, Energy can also be written as

$$\boxed{E = h\nu = \frac{hc}{\lambda}}$$

$$\boxed{p = \left(\frac{h}{\lambda}\right)}$$



Note that  $E = \frac{p^2}{2m}$  is non-relativistic

3 expressions for E

Relativistically:  $p = mv$   
 $E = mc^2$   
 $T = mc^2 - m_0 c^2$   
 $\therefore E = T + m_0 c^2$   
 ②  $E^2 = p^2 c^2 + (m_0 c^2)^2$   
 ( $p = mv$ )

### Relativistic Force

For relativistic motion

$$F = \frac{d}{dt}(mv) = \left(\frac{dm}{dt}\right)v + m\left(\frac{dv}{dt}\right)$$

$$F = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \left(\frac{dv}{dt}\right)$$



21/11/18  
 here we are talking of magnitudes only... no direction

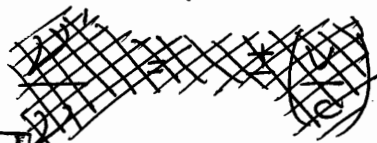
# MECHANICS (17)

- doppler
- aberration
- decay
- density transformation

03/12/2011

## Doppler Effect

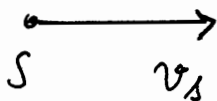
- It is a phenomenon of wave.
- There must be some source of wave and relative motion between source and observer.
- There is change in observed frequency. When they come close frequency increases, & vice versa.



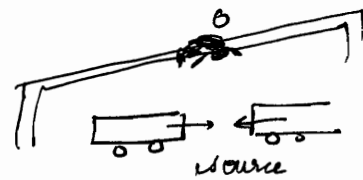
Sound

- This phenomenon is easily observed in sound waves.

[DIRECTION TAKEN FROM SOURCE TO OBSERVER]



$\nu$  : frequency when no relative motion



observer ऊपर से observe करता है

for sound waves इसलिए

0 ऊपर हैं,  
s नीचे हैं!!

$$\nu' = \left( \frac{v - v_o}{v - v_s} \right) \nu$$

where  $v$  is velocity of sound

Here we can see,  $\nu'$  depends on relative velocities between sound and source and sound and observer.

In light, it depends only on  $(v_o - v_s)$

• Hence, in light, doppler effect is the apparent change in frequency of source of wave due to relative motion between source and observer

Longitudinal Doppler Effect

यही ठीक है !!

$$\nu' = \nu \sqrt{\frac{1 + \left(\frac{v}{c}\right)}{1 - \left(\frac{v}{c}\right)}}$$

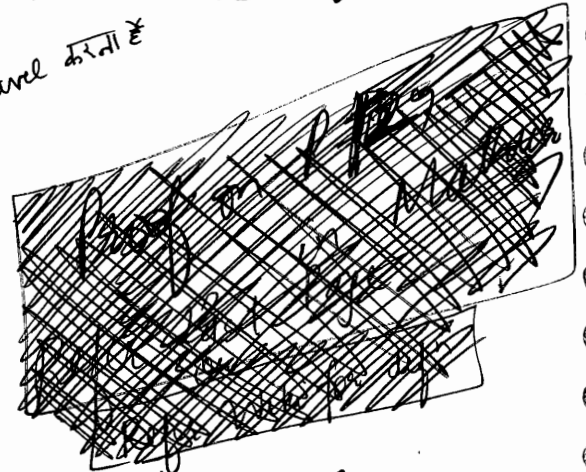
It is a consequence of time dilation.

Let  $v$  be relative velocity between source and observer

$$\nu = \frac{1}{T} = \left(\frac{c}{\lambda}\right)$$

दूरी distance travel करती है  
 वेग velocity है  
 समय time है

This 'T' is proper time



→  $\left(\frac{1}{T}\right)$  is the frequency generated by stationary light bulbs.

Observer will see dilated time if the light source is coming towards observer

$$T' = \frac{T}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}}$$

Hence there are two effects: 1. dilated time और दूरी distance travelled by source.

$$\nu' = \frac{1}{T} \frac{\sqrt{1 + \left(\frac{v}{c}\right)}}{\sqrt{1 - \left(\frac{v}{c}\right)}} = \nu \sqrt{\frac{1 + \left(\frac{v}{c}\right)}{1 - \left(\frac{v}{c}\right)}}$$

- $CT'$ : distance travelled in the mean time in between 2 signals emitted
- $vT'$ : distance that the light has to travel less.



When source comes close, frequency increases

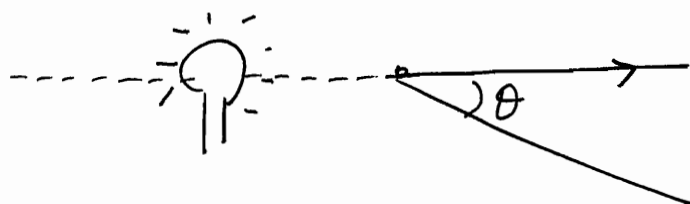
Also note,  $v = c/\lambda$

$$\Rightarrow dy = -\frac{c}{\lambda^2} d\lambda = -\frac{1}{\lambda} v d\lambda$$

$$\Rightarrow \boxed{\frac{dy}{y} = -\frac{d\lambda}{\lambda}}$$

This is called longitudinal doppler effect.

### Transverse doppler effect



~~Take this from last page and copy that formula...~~

Just replace  $v$  by  $v \cos \theta$

When no motion  $v = \frac{c}{\lambda} = \frac{c}{cT} = \frac{1}{T}$

When motion,

$$v' = \frac{c}{\lambda'} = \frac{c}{cT' - vT' \cos \theta}$$

→ Some books may have different form depending on what  $\theta$  we use... whether one seen by source or observer.

$$= \frac{1}{T' \left[ 1 - \frac{v}{c} \cos \theta \right]}$$

General Formula  $v' = v \frac{\sqrt{1 - \frac{v^2}{c^2}}}{\left( 1 - \frac{v \cos \theta}{c} \right)}$

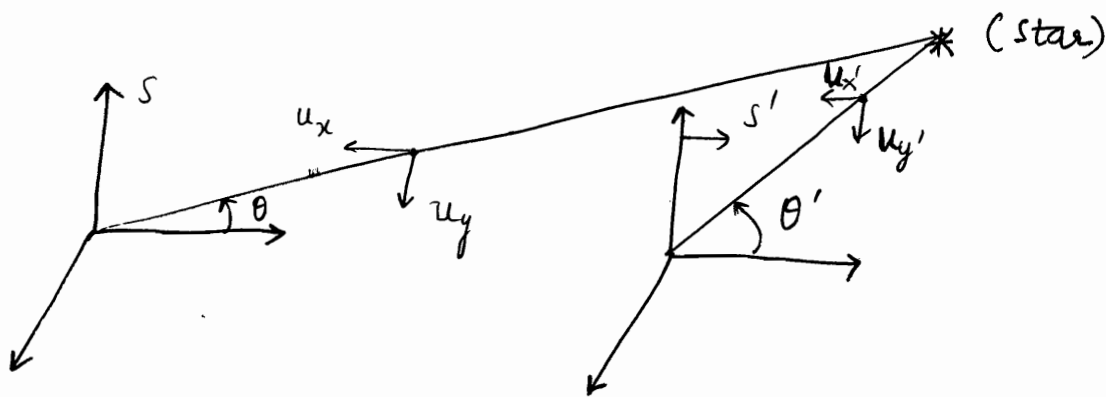
When  $\theta = 90^\circ$

$$v' = v \sqrt{1 - \frac{v^2}{c^2}}$$

This is called Transverse Doppler Effect.

# Stellar Aberration

: The change in apparent position of an object due to motion of the observer..



$$u_x = -c \cos \theta$$

$$u_y = -c \sin \theta$$

$$u'_x = -c \cos \theta'$$

$$u'_y = -c \sin \theta'$$

But we already know

Normal Transformation Equations

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$\& \quad u'_y = \frac{u_y \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{u_x v}{c^2}}$$

$$\left( \frac{u'_y}{u'_x} \right) = \frac{u_y \sqrt{1 - \frac{v^2}{c^2}}}{u_x - v}$$

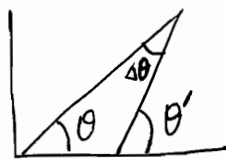
$$v = 3 \times 10^4$$

$$\Rightarrow \tan \theta' =$$

$$\frac{\sin \theta \sqrt{1 - \frac{v^2}{c^2}}}{\cos \theta + \frac{v}{c}}$$

$$\tan \theta \sqrt{1 - \frac{v^2}{c^2}}$$

$$1 + \left( \frac{v}{c \cos \theta} \right)$$



$$\tan(\theta') - \tan \theta = -\frac{v}{c} \frac{\sin \theta}{\cos^2 \theta}$$

(ignore:  $v/c^2 \rightarrow 0$ )

$$\text{Total shift in 6 months} = \left( \frac{2v}{c} \right)$$

$$\begin{aligned} \tan(\theta + \Delta\theta) - \tan \theta &= -\frac{v \sin \theta}{c \cos^2 \theta} \\ \Rightarrow \frac{\sin \Delta\theta}{\cos(\theta + \Delta\theta)} &= -\frac{v \sin \theta}{c \cos^2 \theta} \\ \Rightarrow \Delta\theta &= \frac{v \sin \theta}{c} \end{aligned}$$

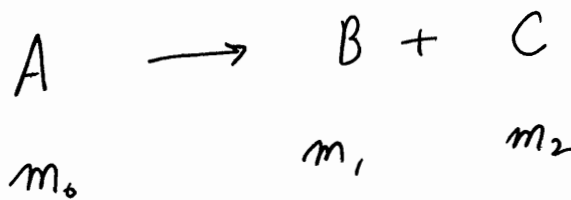
★ James Bradley observed that an overhead star appeared to move in a nearly circular orbit with a period of 1 year with angular diameter of 41" of arc. Also, other stars in other positions have a similar motion in general elliptical. This phenomenon is called Aberration. Its consequence of Earth's motion around the sun.

$$\star \quad \frac{2v}{c} = \frac{3 \times 10^4 \times 2}{3 \times 10^8} \approx 41''$$

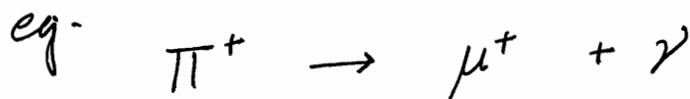
- velocity of light is same in all frames but angle are changed.
- Apparent change in direction  $\theta \rightarrow \theta'$  is called aberration in the path of stars.

## Decay Process

(Derive the expression of  $E_B$  and  $E_C$  in terms of rest masses)



← rest masses  
: decay process  
Simplistic



→ 4-vectors are conserved in decay process

{ since its a spontaneous natural reaction,  $\Rightarrow$  momentum has to be conserved.

↳ Total relativistic energy is conserved

↳ Momentum 4-vector is conserved !!

$$P_\mu = \left( \vec{P}, i \frac{E}{c} \right) \quad : \quad S \text{ frame}$$

$$\vec{P}_B + \vec{P}_C = 0 \quad : \quad \text{Initial Momentum is } 0 \Rightarrow \vec{P}_B = -\vec{P}_C$$

$$\Rightarrow \underline{P_B^2 = P_C^2}$$

Also  $E_A = E_B + E_C$  : relativistic energies

$$\Rightarrow \boxed{m_0 c^2 = E_B + E_C} \quad \text{--- (1) } \checkmark$$

$$\left\{ \begin{array}{l} T_B = E_B - m_1 c^2 \\ T_C = E_C - m_2 c^2 \end{array} \right\} \quad \text{--- (X)}$$

$$E_B^2 = p_B^2 c^2 + m_1^2 c^4$$

$$E_C^2 = p_C^2 c^2 + m_2^2 c^4$$

$$E_B^2 - E_C^2 = [m_1^2 - m_2^2] c^4 \quad \checkmark$$

We know,  $(E_B + E_C)(E_B - E_C) = E_A \cdot (E_B - E_C) = m_0 c^2 (E_B - E_C)$

$$\Rightarrow \boxed{E_B - E_C = \left[ \frac{m_1^2 - m_2^2}{m_0} \right] c^2} \quad \text{--- (2)}$$

From (1) and (2)

$$\Rightarrow E_B = \frac{[m_0^2 + m_1^2 - m_2^2] c^2}{2m_0}$$

$$E_C = \frac{[m_0^2 + m_2^2 - m_1^2] c^2}{2m_0}$$

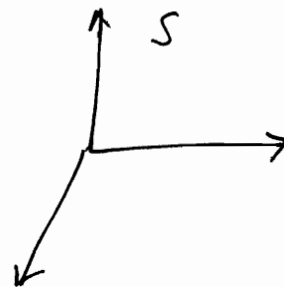
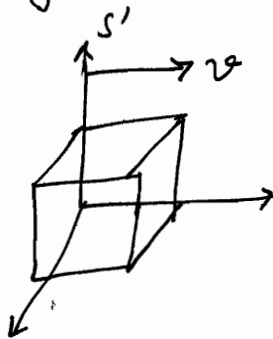
Remember that  $(p^2/2m)$  is an approximate expression for KE, for low velocities its good.

Putting in (X),

$$T_B = c^2 \left[ \frac{(m_0 - m_1)^2 + m_2^2}{2m_0} \right]$$

$$T_C = \frac{c^2 \left[ (m_0 - m_2)^2 - m_1^2 \right]}{2m_0}$$

### Density Transformation



Mass Transformation

$$m' = \left( \frac{m \left( 1 - \frac{u_x v}{c^2} \right)}{\sqrt{1 - \frac{v^2}{c^2}}} \right)$$

where  $m = \frac{m_0}{\sqrt{1 - \frac{u_x^2}{c^2}}}$

In frame \$S'\$ :  $\rho_0 = \frac{m_0}{V_0} = \left[ \frac{m_0}{L^3} \right]$  if  $u_x = 0 \Rightarrow m' = \frac{m_0}{\sqrt{1 - v^2/c^2}}$

in frame \$S\$ :  $\underline{\underline{\rho}} = \frac{m}{V} = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}} L^2 \cdot L \sqrt{1 - \frac{v^2}{c^2}}}$

$$\begin{aligned} L'_x &= L_x \sqrt{1 - \frac{v^2}{c^2}} = \frac{L_0 \sqrt{1 - \frac{v^2}{c^2}}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m_0}{L^3 \left( 1 - \frac{v^2}{c^2} \right)} = \frac{\rho_0}{\left( 1 - \frac{v^2}{c^2} \right)} = \underline{\underline{\alpha \rho_0}} \\ L'_y &= L_y = \underline{\underline{L_0}} \\ L'_z &= L_z = \underline{\underline{L_0}} \end{aligned}$$

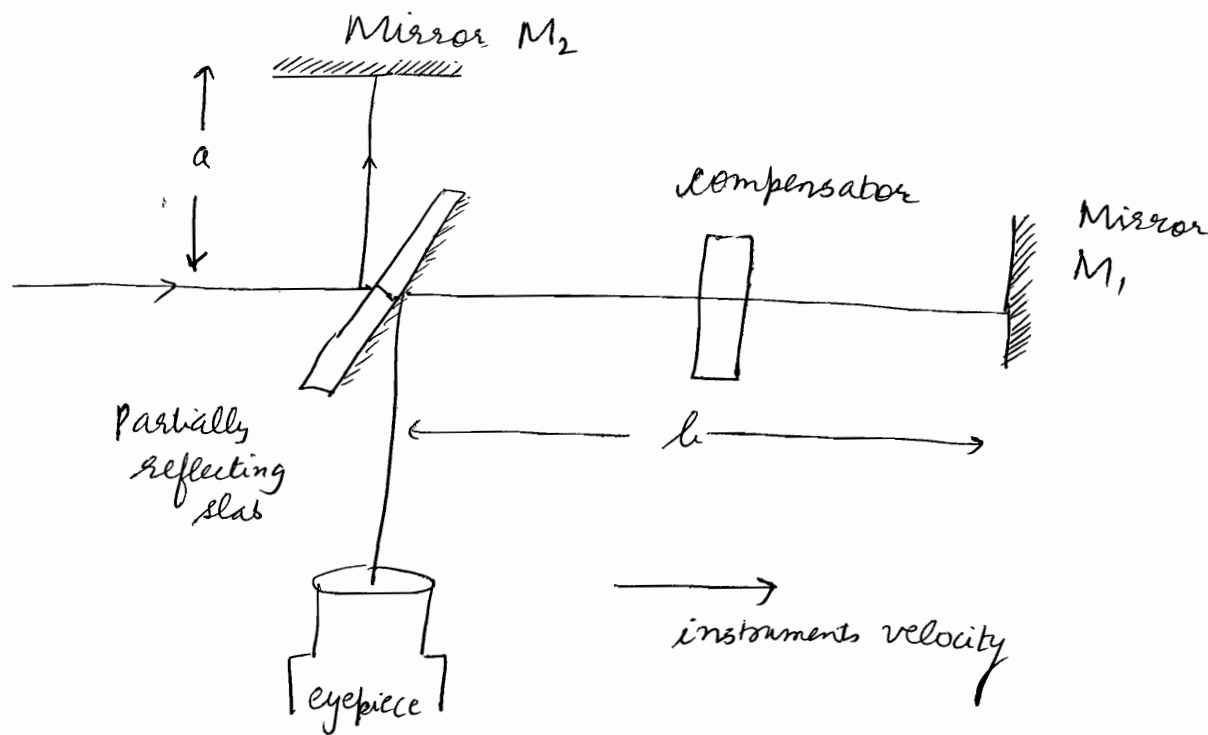
$\Rightarrow$  density increased if measured from relatively moving frame.

## Null result of Michelson Morley Experiment

At the time of the experiment, light was believed to be a wave which travels in a medium called ether. Michelson-Morley Experiment negated the existence of ether.

It used Michelson Interferometer which was based on the premise that due to path difference between the waves travelling in perpendicular directions in a moving ~~exp~~ instrument, there should be a fringe shift. But no such fringe shift was observed.

This result of no-fringe shift in Michelson Experiment is called NULL RESULT. Einstein theory explained these results.



Initially,  $a = b = l$  and instrument is moving at velocity  $v$  w.r.t ether. The time taken by wave is:

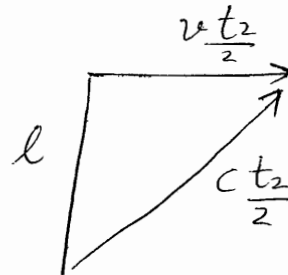
$$\text{Horizontal } t_1 = \frac{l}{c-v} + \frac{l}{c+v} = \frac{2lc}{c^2 - v^2} = \frac{2l}{c} \left(1 - \frac{v^2}{c^2}\right)^{-1} \approx \frac{2l}{c} \left(1 + \frac{v^2}{c^2}\right)$$

## Vertical direction

$$l^2 + \left(\frac{vt_2}{2}\right)^2 = \left(\frac{ct_2}{2}\right)^2$$

$$\Rightarrow \left(\frac{t_2}{2}\right)^2 = \frac{l^2}{c^2 - v^2}$$

$$\Rightarrow t_2 = \frac{2l}{c} \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx \frac{2l}{c} \left(1 + \frac{v^2}{2c^2}\right)$$



do not forget to apply binomials in both cases.

$$\Rightarrow \text{Path difference between horizontal \& vertical waves} = c(t_1 - t_2) = c \cdot \frac{2l}{c} \cdot \frac{v^2}{2c^2} = \left(\frac{lv^2}{c^2}\right) = \Delta$$

So the theory, at the time of experiment, expected a path difference and corresponding fringe shift, but no fringe shift was observed when the complete instrument was rotated (through 90°). Hence this was called NULL RESULT of Michelson Morley Experiment.

### Explanation

[Note that expected fringe shift was expected as the other arm becomes coincident with Earth's velocity, hence path difference in opposite direction  $\Rightarrow$  Fringe shift corresponding to path difference of  $\left(\frac{2lv^2}{c^2}\right)$ ]

① **Fitzgerald** said that length in the direction of light contracts by  $\sqrt{1 - \frac{v^2}{c^2}}$

$$\begin{aligned} \Rightarrow t_1 &= \frac{2l}{c} \left(1 - \frac{v^2}{c^2}\right)^{1/2} \cdot \left(1 + \frac{v^2}{c^2}\right) = \frac{2l}{c} \left(1 + \frac{v^2}{2c^2}\right) \left(1 + \frac{v^2}{c^2}\right) \\ &= \frac{2l}{c} \left[1 + \frac{v^2}{2c^2}\right] = t_2 \end{aligned}$$

② Finally, Einstein came up with bold assumption that velocity of light is constant.

$$\text{i.e. } t_1 = \frac{2l}{c} = t_2$$

Hence, the idea of ether was rejected.

○ Also note  $v = 3 \times 10^4$  m/s. velocity of earth

(u) = point is fixed.

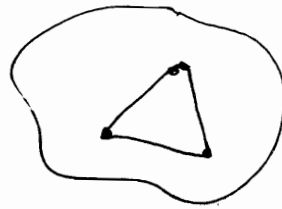
Answer A rigid body is defined by 3 points fixed wrt. each other

Total no. of variables = 6

$(x_2, y_2, z_2)$

$(x_3, y_3, z_3)$

$\therefore (x_1, y_1, z_1)$  are fixed.



$$\begin{aligned} \text{Degrees of Freedom} &= 6 - \text{No. of Constraints} \\ &= 6 - 3 \\ &= 3 \end{aligned}$$

(we can write all equations of constraints if of more marks)

Now even if considered an extra point  $(x_4, y_4, z_4)$

Extra variables are added = 3

Extra constraint added = 3

Hence d.o.f. remain same.



$$\tan\theta = \frac{\sin\theta' \sqrt{1 - v^2/c^2}}{\cos\theta' + v/c}$$

can be derived from the velocity transformation relations.

(2006)

33. A body of rest mass  $m_0$  is moving in the positive y- direction at a velocity of  $0.6 C$  relative to the laboratory frame. Calculate the components of the four dimensional momentum vector in the laboratory frame and in the frame of an observer who is traveling in the positive x- direction at a speed of  $0.8 C$  relative to the laboratory frame.

(2007)

34. State the postulates of the special theory of relativity and based on these obtain Lorentz as well as inverse Lorentz transformations. Hence, obtain an expression to conclude that a moving clock runs more slowly than stationary clock.

(2007)

35. An unstable particle has a lifetime of  $5 \mu s$  in its own frame of reference and is moving towards the earth at a speed of  $0.8 C$ . What will be the lifetime of the particle to an observer on the earth ?

(2007)

36. The length of a moving rod can be defined as the product of its velocity and the time interval between the instants that both the end points of the rod pass a fixed mark in S system. Show that this definition leads to the space contraction

(2008)

37. A meson of rest mass  $\pi$  comes to rest and disintegrates into a muon of rest mass  $\mu$  and a neutrino of zero

rest mass. Show that the kinetic energy of motion of the muon is  $T = \frac{(\pi - \mu)^2 c^2}{2\pi}$

(2008)

38. Show that a four-dimensional volume element  $dx dy dz dt$  is invariant to Lorentz transformation.

(2009)

39. Obtain the relativistic equation for aberration of light using velocity transformation equations.

(2009)

40. What is the significance of the null result of Michelson-Morley experiment? Does it disprove the existence of ether? Justify.

*Proposed that earth drags ether along with it.*

(2010)

16. How is an electron volt connected to other units of energy like the joule or the erg? Determine the speed of an electron of energy 1.3 MeV assuming its rest energy to be .51 MeV. (1998)
17. Prove that the expression  $x^2+y^2+z^2-c^2t^2$  is invariant under Lorentz transformation. (1999)
18. Describe the set-up of Michelson-Morley experiment. Why was a fringe shift expected in it? How are its negative results understood? (1999)
19. What is 'Lorentz Contraction'?  
A spaceship of rest length 100m takes  $4\mu\text{s}$  to pass an observer on Earth. What is its velocity relative to the Earth? (2000)
20. Derive an expression for the mass-energy equivalence using the principle of special relativity. (2000)
21. The mass of muon at rest is 207  $m_e$  where  $m_e$  is the electron rest mass (0.511 MeV). The mean life time at rest for muon is  $2.2\mu\text{s}$ . The life time of muon emerging from an accelerator is measured in the lab as  $6.9\mu\text{s}$ . Estimate the speed of these muons in the laboratory. (2001)
22. A person in a space ship is holding a rod of length of 0.5 m, the space ship is cruising at a speed  $V$  parallel to the earth's surface. What does the person in the space ship notice as the rod is rotated from parallel to perpendicular to the space ship's motion? What does an observer on earth's surface notice? (2001)
23. Two spaceships are moving at a velocity of  $0.9c$  relative to the Earth in opposite directions. What is the speed of one spaceship relative to the other? (2002)
24. An observer A sees two events at the same space point ( $x=y=z=0$ ) and separated by  $t=10^{-6}\text{s}$ . Another observer B sees them to be separated by  $t^1=3\times 10^{-6}\text{s}$ . What is the separation in space of the two events as observed by B? What is the speed of B relative to A? (2002)
25. An observer  $S_1$  sees two bodies A and B having equal rest mass approach each other with equal but opposite velocity of the body  $4c/5$ . To a second observer  $S_2$ , the body A is at rest. What is the velocity of the body B as seen by observer  $S_2$ ? What are the kinetic energies of the body B in the frames of  $S_1$  and  $S_2$ ? (2003)
26. How does Doppler effects of light in relativistic physics qualitatively differ from its non-relativistic analogue? Calculate the Doppler shift in the frequency of a photon traveling along y-axis, with respect to an observer moving along the x-axis with a constant speed  $u$ . (2003)
27. A meson of rest mass  $\pi$  comes to rest and disintegrates to a muon of rest mass  $\mu$  and a neutrino of zero rest mass. Show that the kinetic energy of motion of the muon is  
$$T = (\pi - \mu)^2 c^2 / 2\pi$$
 (2004)
28. Write down the expression for the relativistic mass of a particle moving with a velocity  $v$  in terms of its rest mass. Establish from the above expression Einstein's mass energy relation  $E=mc^2$ . (2004)
30. Show that the length  $L$  of an object moving with a velocity  $v$  is given in the direction of motion by  
$$L = L_0(1 - v^2/c^2)^{1/2}$$
  
Where  $L_0$  is the proper length and  $c$  is the velocity of light in free space.  
What will be the shape of a spherical ball while moving under relativistic regime? *Prolate spheroid*
31. Prove that two successive Lorentz transformations are equivalent to another Lorentz transformation. Hence write down the Einstein's velocity addition relation. (2006)
32. The source  $S$  moves along the  $x'$ -axis at a speed  $v$ , and emits light at an angle  $\theta'$  to the  $x'$ -axis of its own frame. In the  $S$ -frame the emitting angle with the  $x$ -axis is  $\theta$ . Hence  $x$  and  $x'$ -axis are coincident. Show that the exact relativistic aberration formula.

\* relative increase in a variable  $a = \left(\frac{da}{a}\right)$

**TUTORIAL SHEET: 6****Special Relativity**

1. Show from the Lorentz transformation that two events ( $t_1=t_2$ ) at different points ( $x_1 \neq x_2$ ) in reference frame S are not in general simultaneous in reference frame S<sup>1</sup> which is moving in the +x direction with the constant velocity V with respect to S. (1988)
2. What is the momentum of a proton having kinetic energy 1 Be V? The energy equivalent to proton rest mass is 938 Mev. (1988)
3. Write down Lorentz transformation relations and prove that  $x^2+y^2+z^2-c^2t^2$  is invariant under this transformation. (1990)
4. An event occurs at  $x^1=60\text{m}$ , at  $t^1=8 \times 10^{-8}\text{ s}$  in a reference frame S<sup>1</sup> which is moving along the common X or X<sup>1</sup> axis with a speed of  $3c/5$  with reference to a stationary frame S. The origins of two frames coincide at  $t=0, t^1=0$ . Deduce the space time coordinates of the event in the frame S. (1990)
5. Prove that addition theorem of velocities in special theory of relativity. Two bodies A&B are moving away in opposite directions, each with a speed of  $0.70C$  with respect to a stationary observer. Deduce the speed of B as measured by A? (1990)
6. The half life of  $\mu$ -meson at rest is  $2 \times 10^{-8}$  seconds. Determine the half life of  $\mu$ -meson while traveling with half the speed of light in vacuum. (1992)
7. Write down the expression for the dependence of mass of a particle on its velocity in special relativity. What will be the speed of a particle if its mass becomes double of its rest mass. (1992)
8. State Lorentz transformation equations. Show that for  $V/c \ll 1$ , the Lorentz transformation equations reduces to Galilean transformation equations. Explain the physical significance of Lorentz transformations. (1993)
9. State and explain variation of mass with velocity, hence find an expression for the density of a body in an arbitrary inertial frame of reference. (1994)
10. The density of gold in its proper frame of reference is  $19.3 \times 10^3 \text{ kg/m}^3$ . Determine its density in a frame of reference where its velocity is  $0.9C$ , C is the velocity of light in free space. (1995)
11. In the relativistic region, compare the relative increase in velocity with the relative increase in energy of a particle. (1995)
12. Determine the speed of an electron, which has kinetic energy 1 Mev. The energy equivalent to the rest mass of an electron is 0.51 MeV. (1995)
13. The coordinates of an event in an inertial frame S are  $(25\text{m}, 0, 0, 5 \times 10^{-8}\text{S})$ . What will be the coordinates of this event in another inertial frame S<sup>1</sup> moving with a velocity  $0.6C$  in +x direction with respect to S. The origin of two frames coincide at  $t=t^1=0$ . (1996)
14. Obtain the relativistic transformation relation for density in inertial frames. What is the equivalent energy corresponding to 1 amu of mass? (1997)
15. A  $\mu$  meson travels towards the earth's surface from high up in the atmosphere with a speed of  $0.99C$ . It decays after traveling a distance of 6km. In what time does the  $\mu$  meson decay as measured by observers in reference frames (i) bound to the earth (ii) bound to the meson itself. (1998)

**TUTORIAL SHEET: 5**  
**Mechanics of Continuous media**

1. Show that the Bulk modulus  $K$ , Young's modulus  $Y$  and Poisson's ratio  $\sigma$  are connected by the relation

$$K = \frac{Y}{3(1-2\sigma)}$$

(2008)

2. What do you understand by streamline motion and critical velocity of a viscous liquid through a capillary tube. Capillaries of lengths  $l$ ,  $2l$  and  $\frac{l}{2}$  are connected in series. Their radii are  $r$ ,  $\frac{r}{2}$  and  $\frac{r}{3}$  respectively. If the streamline flow is maintained and the pressure across the first capillary is  $P_1$ , deduce the pressures across the second and the third capillaries.

(2008)

3. Show that the total energy per unit mass of liquid flowing from one point to another without any friction remains constant throughout the displacement.

(2009)

\*  $I_{\text{solid sphere}} = \frac{2}{5} m r^2$

$I_{\text{hollow sphere}} = \frac{2}{3} m r^2$

① Tut 3A  
Central Force Motion  
Mechanics of  
Continuous Media

**Tutorial Sheet: 4**  
**Rigid Body Dynamics**

1. Explain precessional motion of a top. A solid sphere of radius 2 cm and mass 50gm has a thin nail of length 5mm fixed perpendicular to its surface. When this sphere spins like a top with a speed of 20 rev/sec. What will be its precessional speed (1995)
2. Explain the precession of a spinning top and show that precessional velocity is independent of the angle of inclination. A top is spinning with 20 rev/s about an axis inclined at  $30^\circ$  with the vertical. Its radius of gyration is 5 cm. The center of mass is  $\pi$  cm from the pivot point. Calculate the frequency of precession. (1999)
3. Starting from the definition of angular momentum of a single particle, obtain an expression for the angular momentum of a rotating rigid body. Hence, discuss the time derivative of the angular momentum and deduce the law of conservation of momentum. (1999)
4. Show that the combined effects of translation of the center of mass and rotation about an axis through the center of mass are equivalent to a pure rotation with the same angular speed about an axis through the point of contact of a rolling body. (2000)
5. Write the Euler's equations for the rotational motion of a rigid body with one point fixed, under the action of a torque  $N$ . Apply these equations to discuss the rotational motion of a symmetrical top on the absence of any force other than the reaction at the fixed point.
6. What do you mean by the moments and products of inertia? Show that the angular momentum vector is related to the angular velocity components by linear transformation relations? (2004)
7. Derive Euler's equations of motion for a rigid body rotating about a fixed point under the action of a torque. When a rigid body is not subjected to any net torque, write down Euler's equations of motion of the body with one point fixed. (2006)
8. The angular momentum  $\vec{M}$  of a rigid body comprising of  $N$  particles and rotating with angular velocity  $\vec{\omega}$  is given by  $\vec{M} = \sum_{k=1}^N m_k \vec{r}_k \times (\vec{\omega} \times \vec{r}_k)$  where the origin coincides with the centre of mass. Express the components of  $\vec{M}$  in terms of components of the inertia tensor. Hence, show that the most general free rotation of a spherical top is a uniform rotation about an axis fixed in space. (2007)
9. Derive an expression for the moment of inertia of a rigid body about any axis. What is an "ellipsoid of inertia"? Explain clearly what you mean by the terms "principal axes" and "principal moments of inertia"?  
Find the moment of inertia of a thin rectangular lamina about an axis passing through the centre of the lamina and perpendicular to its plane. Hence determine the moments of inertia about axes passing through the midpoints of its both sides and perpendicular to its plane. (2008)
10. Show that for any rigid body consisting of at least three particles not arranged in one straight line, number of independent degrees of freedom is six.  
Define Euler's angles  $\theta$ ,  $\phi$  and  $\psi$  to describe the configuration of such a rigid body.  
Consider two frames of reference, one fixed to the body and the other to the space defined as  $S' = (x', y', z')$  and  $S = (x, y, z)$  respectively, Show that the angular momentum ( $\vec{L}$ ) of the rigid body in the two frames are related by  
$$\left(\frac{d\vec{L}}{dt}\right)_S = \left(\frac{d\vec{L}}{dt}\right)_{S'} + \vec{\omega} \times \vec{L}$$
Where  $\vec{\omega}$  is the angular velocity of rotation. (2009)

**TUTORIAL SHEET: 3**

**Gravitation and Central Force Motion**

- 1 State Kepler's laws of planetary motion. Assume the law of gravitation to be of the form  $F = \frac{GMm}{r^n}$  where n is some number. Find the value of n which will be consistent with Kepler's third law. For this you may assume planetary orbits to be circular. (1990)
- 2 A Satellite moves round the earth at a distance of  $3.884 \times 10^5$  km from its Centre. Find its period of revolution. in days. Proceed to deduce the distance for a geo- stationery satellite. (1990)
- 3 A body falls from a great height towards the surface of the moon. Write down its equation of motion and solve it to determine the speed with which the body strikes the surface. Also find its numerical value. (1991)
- 4 Deduce the magnitude and direction of the acceleration of the moon at new moon. (1991)
- 5 State Kepler's laws of planetary motion. Why does a missile need an escape velocity to escape from earth? (1992)
- 6 Calculate the speed of a satellite orbiting the planet Jupiter at a distance 108 Km, above its surface (The mass and radius of Jupiter are  $1.9 \times 10^{27} \text{kg}$  and 69892 Km respectively) (1992)
- 7 A Satellite, revolving in a circular equatorial. Orbit at height  $1.36 \times 10^4$  km from earth's surface from west to west, appears over some spot at the equator every 6 hrs. Show that the data are consistent with known earth's radius  $6.4 \times 10^3 \text{ km}$  and  $g = 9.8 \text{ m/S}^2$  at earth's surface (1993)
- 8 State Kepler's laws and prove that for elliptical orbits the square of the period of a satellite is proportional to the cube of the semi- major axis. (1995)
- 9 A satellite is launched into a circular orbit close to the surface of the earth. Find an expression for the additional velocity that has to be "imparted" to the satellite to overcome the gravitational pull. (1995)
- 10 A satellite has apogee and perigee heights above the surface of earth as 4000km and 640km respectively calculate the eccentricity of the orbit and semi-major axis. Radius of the earth is 6400 Km. (1997)
- 11 A particle is moving under a central force field. Obtain the equations of motion and prove that in such a field the aerial velocity of the particle remains constant. (1997)
- 12 Moving in a closed elliptical orbit, Haley's comet goes round the sun once every 75 years. At its point to closes approach its distance from the sun is  $8.9 \times 10^7 \text{ km}$ . If mass of the sun is  $2 \times 10^{30} \text{ kg}$ , what will be the greatest distance of the comet from the sun (1997)
- 13 Prove that the path of a particle moving in a central force field is a plane curve and that the angular momentum of the particle remains constant in time. (1998)
- 14 Neptune's period of revolution is 165 yrs. What is the average radius of Neptune's orbit in terms of the Sun- Earth distance ? (2000)
15. Two bodies of masses  $M_1$  and  $M_2$  are placed at a distance d apart. Show that at this position where the gravitational field due to them is zero, the potential is given by

$$V = -\frac{G}{d} (M_1 + M_2 + 2\sqrt{M_1 M_2})$$

$G = 6.67 \times 10^{-11} \text{ S.I. units}$

(2009)

To solve  $\ddot{x} = -\frac{A}{x^2}$   
 Multiply by  $2\dot{x}$  on both sides  
 It becomes  $d(\dot{x}^2) = A d(\frac{1}{x})$

**TUTORIAL SHEET: 2**  
**Rotating Frames of Reference**

1. ✓ What is understood by the term 'Coriolis force'? Obtain expressions for velocity & acceleration of a particle in rotating coordinate system. (1988)
2. ✓ Define Coriolis force and write an expression for it, through suitable examples, explain the way this force varies in different parts of the earth's surface and for different velocities of the concerned particle. (1989)
3. ✓ Explain what Coriolis force means. Discuss the action of Coriolis force on a body falling freely on the earth at latitude ' $\lambda$ '. (1994)
4. ✓ If the earth were to rotate at ' $n$ ' times its present speed of rotation about its axis, the apparent weight of a body at the equator would assume zero value. Find an expression for ' $n$ '. (1995)
5. ✓ Define Coriolis force. Obtain an expression for the Coriolis force. How does it account for the whirling of winds in opposite directions in Northern and southern Hemispheres? (1996)
6. ✓ Obtain the equation of motion of a particle moving relative to a rotating frame of reference Explain the term representing Coriolis force in this expression. (2001)
7. ✓ For a freely falling body from the height ' $h$ ' on the surface of the earth in the northern hemisphere with a latitude ' $\theta$ ', show that the deviation of the body towards east at the final stage is given by  $\frac{1}{3} \omega \cos\theta (8h^3/g)^{1/2}$ , where  $\omega$  is the angular velocity of the earth and ' $g$ ' is the acceleration due to gravity. (2004)
8. a. ✓ Derive the relation  $\vec{V} = \vec{V}_0 + \omega \times \vec{r}$ , where  $\vec{V}$  is the velocity of a particle located at  $\vec{r}$  in a fixed frame of reference S and  $\vec{V}'$  that observed in frame S' rotating with angular velocity  $\vec{\omega}$  with respect to S but having the common origin. (2007)
- b. ✓ Show using the above relation that the equation of Motion of the particle in S gets modified in S' giving rise to Various fictitious force. Identify the coriolis force and describe its effect on the flow of rivers. (2007)

20) Define scattering cross-section. A charged particle of mass  $m$  and charge  $Ze$  is scattered by another charged particle of charge  $Ze$  at rest. Deduce the expression for the scattering cross-section.

(2000)

21) Using the rocket equation and its integral, find the final velocity of a single stage rocket. Given that (a) the velocity of the escaping gas is 2500 m/s (b) the rate of loss of mass is  $(m/200)/\text{sec}$ . (where  $m$  is the initial mass and  $0.27m$  is the final mass.)

(2002)

22) Derive the relationship between the impact parameter and the scattering angle for the scattering of an alpha particle of charge  $+2e$  by a nucleus of charge  $+Ze$ .

Calculate the impact parameter for an angle of deflection of  $30^\circ$  if the kinetic energy of the alpha particle is  $6 \times 10^{-13}$  joules.

(2002)

25) If a single stage rocket fired vertically from rest at the earth's surface burns its fuel in a time of 30 sec and their relative velocity  $v_r = 3 \text{ km Sec}^{-1}$ , what must be the mass ratio  $m_0/m$  for a final velocity is of  $8 \text{ km/sec}$ ?

(2004)

26) Considering the scattering of  $\alpha$ -particles by the atomic nuclei, find out the Rutherford scattering cross-section. Explain the physical significance of the final expression.

(2005)

27) Derive an equation of motion for a variable mass. Explain how it is applied in the motion of a rocket.

(2006)

28) What is center of mass? Show that the total linear momentum of a system of particles about the centre of mass is zero.

(2006)

29) A force field is given by

$$\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}.$$

Is it a conservative field? If so, what is the scalar potential?

(2008)



- 10) A radioactive nucleus of mass  $m_0$  amu emits alpha particle with kinetic energy  $E_\alpha$ . If the disintegration occurs when the nucleus is free, deduce an expression for the energy evolved ( $E_T$ ) during the disintegration. (1993)
- 10) Find the fractional decrease of kinetic energy of a of mass  $m_1$  when a head on elastic collision takes place with another particle of mass  $m_2$  initially at rest. In this context show why hydrogen would be best to be used for slowing down. Actually  $D_2O$ , not  $H_2O$  is used. Why? (1993)
- 11) What is centre of mass? Show that there exists only one centre of mass in a system of particles. Discuss the usefulness of centre of mass in studying motion of a system of particles. (1994)
- 12) The distance between the centres of Oxygen and Carbon atoms in a CO molecule is  $1.2 \text{ \AA}$ . Determine the position of the centre of mass of the molecule relative to Carbon. (Assume atomic masses of Carbon and Oxygen as 12 & 16 respectively). (1994)
- 13) A particle of mass  $m$ , moving with an initial velocity  $V_0$  is acted on by a central repulsive inverse square force,  $F = \frac{k}{r^2}$ . Show that the Scattering angle  $\theta$  depends on the impact parameter 'b' as
- $$\cot\left(\frac{\theta}{2}\right) = \frac{mv_0^2}{k} b \quad (1990)$$
- 14) A particle of mass  $m_1$  moving with a velocity  $V_1$  undergoes an elastic collision with a particle of mass  $m_2$  at rest, in laboratory - frame. After the collision the first particle moves at a certain angle to the direction of its initial velocity, and this angle is  $\theta$  in laboratory-frame and  $\phi$  in centre of mass-frame. If the ratio of masses  $\frac{m_2}{m_1}$  is A, show that  $\theta$  &  $\phi$  are related as  $\tan \theta = \frac{\sin \phi}{\cos \phi + \frac{1}{A}}$  (1996)
- 15) In the  $NH_3$  molecule, the three hydrogen atoms form an equilateral triangle. The distance between the centre of this triangle from each hydrogen atom is  $0.939 \text{ \AA}$ . The nitrogen atom is at the apex of the pyramid with the three hydrogen atoms forming the base. The distance between the hydrogen and nitrogen atoms is  $1.014 \text{ \AA}$ . Find the position of the centre of mass relative to the nitrogen atom. (1997)
- 16) How do we infer the law of conservation of linear momentum from Newton's laws of motion? A stationary bomb explodes and on explosion, it fragments into three parts. Two of these parts, which are of equal masses, fly apart perpendicular to each other with a velocity of  $60 \text{ m/s}$  each. The third part has a mass four times the other two. Find the magnitude and the direction of the velocity of the third part (1999)
- 17) Consider the motion of a rocket in a gravitational field and derive an expression for its final velocity when the fuel burns at a constant rate till it is fully consumed. (1999)
- 18) The mass of the moon is about 0.13 times the mass of the earth. The distance from the center of the moon to the center of the earth is 60 times the radius of the earth. Taking the earth's radius to be  $6378 \text{ km}$ , find out the distance of the center of mass of the earth-moon system from the center of the earth. (2002)
- 19) Ram and Shyam are two skaters weighing  $40 \text{ kg}$  and  $60 \text{ kg}$  respectively. Ram traveling at  $4 \text{ m/s}$  meets Shyam traveling at  $2 \text{ m/s}$  in opposite direction and collides head on.
- If they remain in contact, what is their final velocity?
  - How much kinetic energy is lost?
- (2000)

**CIVIL SERVICES EXAMINATION (MAINS)**

**PHYSICS PAPER - I: MECHANICS**

**TUTORIAL SHEET: 1**

**Conservation laws**

1) What is the recoil energy in electron - volts of mass  $10^{-23}$  gm after emission of a  $\gamma$  ray of energy of 1 Mev? (1990)

2) Define differential scattering cross-section. Write down the dependence of Rutherford scattering cross-section  $\sigma(\theta)$ , on the scattering angle  $\theta$  and sketch this dependence graphically. In the present case the total scattering cross section  $\sigma = \int \sigma(\theta) d\Omega$  turns out to be infinite. Comment on this result. (1990)

3) A neutron of energy 1 MeV collides with a stationary helium nucleus and is scattered. Deduce the momentum of the neutron and of the helium nucleus in their center of mass system. (1990)

4) Define differential scattering cross section for a scattering process. The differential scattering cross-section for neutrons scattered elastically from a solid is of the form

$$\sigma(\theta) = A e^{-B(\vec{k}_i - \vec{k}_f)^2}$$

where A & B are constants and  $\vec{k}_i$  and  $\vec{k}_f$  are respectively the wave vectors of the incident and scattered neutron. Determine the total Scattering cross - sections, given  $|\vec{k}_i| = |\vec{k}_f|$ . (1991)

5) Prove that if E and  $E^1$  are respectively the neutron energies in the laboratory system, before and after collision with a nucleus of mass number A, then

$$\frac{E^1}{E} = \frac{1 + A^2 + 2A \cos \theta}{(1 + A)^2}$$

Where  $\cos \theta$  is the cosine of the scattering angle in the centre of mass system. (1991)

6) What do you mean by centre of mass of a system of particles? Derive expressions for the instantaneous position vector and velocity of the centre of mass of such a system of particles (1992)

7) Using Rutherford's observation that the number of  $\alpha$  - particles scattered at angle  $\phi$  and falling on unit area of the screen varied as  $\left[ \operatorname{cosec} \left( \frac{\phi}{2} \right) \right]^4$ , deduce an expression for the probability of scattering between angles  $\phi$  &  $\phi + d\phi$ . (1992)

8) A rocket of mass 1000 kg. is ready for a vertical take off. The exhaust velocity of its fuel is 4.5 km/s. Deduce  
 (a) The minimum rate of fuel ejection so that the rocket weight be just balanced  
 (b) The velocity acquired in 8 seconds if the fuel ejection rate is 2.50 kg/s. (You may neglect the effect of changing mass of the rocket in the given conditions). (1993)

1

*thrust force*

$$\star F_{net} = \underbrace{u_{rel} \left( \frac{dm}{dt} \right)}_{\text{thrust force}} - mg$$

$$= m a_{net}$$

pulls apart & released. Show that k.E. of blocks are, at any time, inversely proportional to masses.

Ans  $m_1 v_1 + m_2 v_2 = C$        $k_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 \frac{C^2}{m_1^2} = \left(\frac{C^2}{2m_1}\right) = \alpha \left(\frac{1}{m_1}\right)$

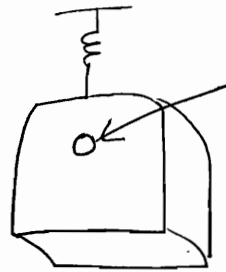
Q The wire cage of a bird is suspended from a spring balance. How does the reading on balance differ when the bird flies off from that when it just sits quietly.

Ans depends on structure of cage



: reading drop

Cage



A very small key hole for bird to survive  
: no change in reading

fully covered metallic

○ As consequence of fluttering, air pushes the bird up and balanced its weight. The bird pushes the air downwards. Downward force so created is transmitted to ground and not to cage as cage has no solid base.

Q Calculate loss (as a %) in k.E. of neutron of mass  $m_1$ , when it strikes stationary nucleus of  $m_2$  mass initially at rest. Energy transferred to  $m_2$  can be increased by including another body of mass  $m_3$  in between.  $[m_1 \rightarrow m_3 \rightarrow m_2]$   
Show that energy transferred is maximum for  $m_3 = \sqrt{m_1 m_2}$

(i)  $v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) u_1$        $v_2 = \left(\frac{2m_1}{m_1 + m_2}\right) u_1$

$\frac{\Delta kE_2}{(k.E)_1} = \left(\frac{\frac{1}{2} m_2 v_2^2 - \frac{1}{2} m_2 u_2^2}{\frac{1}{2} m_1 u_1^2}\right) = 1 - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 = \frac{4m_1 m_2}{(m_1 + m_2)^2}$

(ii)  $E_{\text{gained}} = E_1 \left(\frac{4m_1 m_2}{(m_1 + m_2)^2}\right)$ , for an intermediate particle  $m_3$ ,

$E_{\text{gained by 2}} = E_1 \frac{4m_1 m_3}{(m_1 + m_3)^2} \cdot \frac{4m_3 m_2}{(m_3 + m_2)^2} = E_0 \Rightarrow \frac{dE_0}{dm_3} = 0$

also  $\lambda \equiv \sqrt{E_0} = \sqrt{\alpha \left[\frac{m_3}{(m_1 + m_3)(m_2 + m_3)}\right]}$  ;  $\frac{d\lambda}{dm_3} = 0 \Rightarrow \boxed{m_3 = \sqrt{m_1 m_2}}$

① A motion is called bounded if  $r$ ,  $\dot{r}$  vanishes at extreme values of  $r$ , say

$$\underline{r = r_{\max}} \quad \text{and} \quad \underline{r = r_{\min}}$$

② Orbit is closed if particle retraces its path.

③ Orbit is stable st. at  $r = r_0$ ,

$$\underline{\frac{dV_{\text{eff}}}{dr} = 0}, \quad \underline{\frac{d^2V_{\text{eff}}}{dr^2} > 0}$$

Using above 2 conditions (take  $V = a r^{n+1}$ )  
Orbit is stable & closed  
only for  $n = 1, -2$   
 $T_0 = \frac{b}{r^2}$

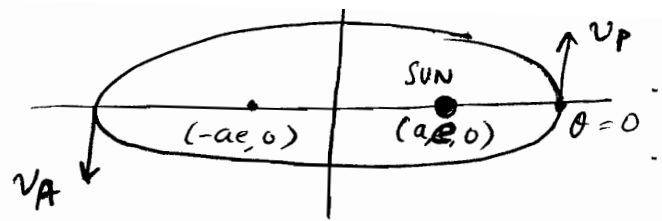
$V = \frac{k}{r}$  or  $k r^2$  type forces  
ie.  $F = -\frac{k}{r^2}$  (inverse square law) or  $-kx$  (hook's law)

⊛ Note that in planetary motion,

$$\frac{l}{r} = 1 + e \cos \theta$$

$r$  is measured from sun which is located at focus of ellipse.

$$r = \frac{l}{1 + e \cos \theta}$$



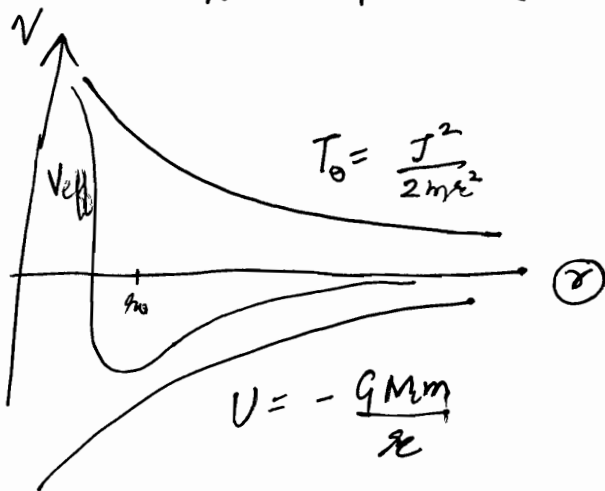
$$\theta = 0 \quad r_p = \frac{l}{1 + e} \quad \text{Perigee}$$

$$\theta = \pi \quad r_A = \frac{l}{1 - e} \quad \text{Apogee}$$

$$m v_p r_p = m v_A r_A$$

$$\Rightarrow \boxed{v_p r_p = v_A r_A}$$

$$\Rightarrow \frac{v_p}{v_A} = \frac{r_A}{r_p} = \left( \frac{1 + e}{1 - e} \right) \quad \checkmark$$



$E < 0$  : bound motion

$E > 0$  : particle escapes

$$E = \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{J^2}{2mr^2} + V(r)}_{V_{\text{eff}}}$$

Proof of 3.1

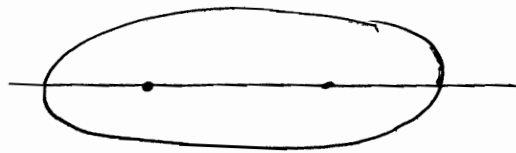
For circular motion,

$$\frac{GMm}{r^2} = m\omega^2 r$$

$$T = (2\pi/\omega) = 2\pi \sqrt{\frac{m r^3}{GMm}} = \frac{2\pi}{\sqrt{GM}} r^{(3/2)}$$

For elliptical motion,

$$l = \frac{J^2}{k\mu} = \frac{b^2}{a}$$



$$T = \frac{\pi ab}{\left(\frac{J}{2\mu}\right)}$$

$$T = \left( \frac{\text{Area swept}}{\text{Areal velocity}} \right)$$

~~~~~

$$b = \frac{J \sqrt{a}}{\sqrt{k\mu}}$$

$$\Rightarrow T = \frac{\pi a}{J} \frac{J \sqrt{a}}{\sqrt{k\mu}} 2\mu$$

$$= \frac{2\pi \mu}{\sqrt{k\mu}} a^{(3/2)}$$

$$\frac{b^2}{a} = l$$

$$T = \left( \frac{2\pi}{\sqrt{GM}} \right) a^{3/2}$$

$$[\mu = m]$$

⇒ circle or ellipse

For circular case,

$$\frac{2EJ^2}{\mu k^2} = -1$$

$$\Rightarrow E = -\frac{\mu k^2}{2J^2} = -\left(\frac{GMm}{2r}\right) \quad \text{[only for circular case]}$$

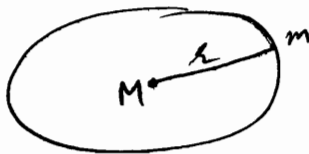
$$\frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{\mu k^2}{2J^2}$$

$$\frac{1}{2}mv^2 = \frac{GMm}{r} - \frac{GMm}{2r}$$

$$\frac{1}{2}mv^2 = \frac{GMm}{2r}$$

$$\Rightarrow v_0 = \sqrt{\frac{GM}{r}} = (v_e/\sqrt{2}) \quad \text{In circular orbit}$$

↑  
Vorbital =  $\frac{v_{\text{escape}}}{\sqrt{2}}$



$$\frac{GM}{r^2} = \omega^2 r$$

∴  $\vec{r} = 0$

$$\Rightarrow GM = \omega^2 r^3$$

$$E = -\frac{\mu k^2}{2J^2} = -\frac{m G^2 M^2 m^2}{2m^2 \omega^2 r^4}$$

$$\Rightarrow E = -\frac{m G^2 M^2}{2\omega^2 r^4}$$

$$= -\frac{m \omega^4 r^6}{2\omega^2 r^4}$$

$$= -\frac{m \omega^2 r^2}{2}$$

$$= -\left[\frac{GmM}{2r}\right]$$

$$\frac{1}{2} \mu v^2 < \left( \frac{k}{r} \right)$$

$$\Rightarrow \frac{1}{2} m v^2 < \frac{GMm}{r}$$

$$\Rightarrow \boxed{v < \sqrt{\frac{2GM}{r}}}$$

At the verge of escape,  $E_{\text{net}} = 0$

$$\Rightarrow \boxed{v_{\text{esc}} = \sqrt{\frac{2GM}{r}}}$$

Escape velocity : velocity when planets is just able to ward off attraction of bigger mass

✓ If  $E = 0$  i.e. escape velocity

$\Rightarrow e = 1$  i.e. it will follow a parabolic path.

✓ If  $E > 0 \Rightarrow e > 1$  i.e. hyperbolic path.



$$\frac{J/k\mu}{r} = 1 + \sqrt{1 + \frac{2EJ^2}{\mu k^2}} \cos(\theta - \theta_0)$$

$$l = \left( \frac{J^2}{k\mu} \right)$$

Hemu का भार  
KEMU

$$\text{always} \\ \frac{\langle J^2 \rangle}{\langle \mu \rangle}$$

$$e = \sqrt{1 + \frac{2EJ^2}{\mu k^2}}$$

&

$$\frac{\langle \alpha \rangle}{\langle k^\alpha \rangle}$$

For ellipse,  $e < 1$   
 $\Rightarrow E$  is negative

$$E = \frac{1}{2} \mu v^2 - \frac{k}{r}$$

$$\approx \frac{1}{2} m v^2 - \frac{k}{r}$$

[ for planetary motion ]

✓ if  $k \cdot E > \text{Potential Energy}$   
 $\Rightarrow$  planet will escape

Hence  $k \cdot E < \text{Potential Energy}$

$$\Rightarrow \frac{du}{d\theta} = -A \sin(\theta - \theta_0)$$

$$\Rightarrow \frac{J^2}{2\mu} \left[ \frac{k^2 \mu^2}{J^4} + A^2 \cos^2(\theta - \theta_0) + \frac{2k\mu A \cos(\theta - \theta_0)}{J^2} + A^2 \sin^2(\theta - \theta_0) \right]$$

$$= E + \frac{k^2 \mu}{J^2} + kA \cos(\theta - \theta_0)$$

$$\Rightarrow A = \frac{k\mu}{J^2} \sqrt{1 + \frac{2EJ^2}{\mu k^2}}$$

Poora cancel-karke  
 krke derive krte!!  
 Just show 2-3 steps and  
 write expression of A from  
 the result.  
 i.e. we have got  $\frac{(J^2/k\mu)}{r} = 1 + \frac{(J^2 A)}{(k\mu)} \cos(\theta - \theta_0)$

$$\Rightarrow A = \frac{k\mu}{J^2} \sqrt{1 + \frac{2EJ^2}{\mu k^2}}$$

$$\Rightarrow u = \frac{k\mu}{J^2} + \frac{k\mu}{J^2} \sqrt{1 + \frac{2EJ^2}{\mu k^2}} \cos(\theta - \theta_0)$$

$$\frac{1}{r} = \frac{k\mu}{J^2} \left[ 1 + \sqrt{1 + \frac{2EJ^2}{\mu k^2}} \cos(\theta - \theta_0) \right]$$

$$\Rightarrow \frac{[J^2/k\mu]}{r} = 1 + \sqrt{1 + \frac{2EJ^2}{\mu k^2}} \cos(\theta - \theta_0)$$

$$E = \frac{1}{2} \mu v^2 + U = \frac{1}{2} \mu v^2 - k u$$

$$U = - \int F \cdot dr$$

$$= + \int \frac{k}{r^2} dr = - \frac{k}{r} = -k u$$

~~$$E + k u = \frac{1}{2} \mu v^2 = \frac{1}{2} \mu \left[ \dot{r}^2 + r^2 \dot{\theta}^2 \right]$$

$$= \frac{1}{2} \mu \left[ \frac{J^2}{\mu^2} \left( \frac{du}{d\theta} \right)^2 + \frac{1}{u^2} \frac{J^2 u^4}{\mu^2} \right]$$

$$= \frac{1}{2} \frac{\mu J^2}{\mu^2} \left[ u^2 + \left( \frac{du}{d\theta} \right)^2 \right]$$~~

$$v^2 = \frac{J^2}{\mu^2} \left[ u^2 + \left( \frac{du}{d\theta} \right)^2 \right]$$

$$v^2 = v_r^2 + v_\theta^2$$

$$= \dot{r}^2 + r^2 \dot{\theta}^2$$

$$k \cdot E = \frac{J^2}{2\mu} \left[ u^2 + \left( \frac{du}{d\theta} \right)^2 \right] = E + k u$$

Now we can express A in terms of E and J.

$$F(r) = -\frac{k}{r^2} = -ku^2$$

$$k: \oplus ve$$

$k = G M m \Rightarrow$  Planetary Motion (attractive)

$F(r) = \frac{k}{r^2}$  &  $k = +\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0} \Rightarrow$  Rutherford Scattering Electrical Field  
(repulsive)

$$u + \frac{d^2 u}{d\theta^2} = +\frac{\mu k}{J^2}$$

$$\frac{d^2 u}{d\theta^2} \left[ u - \frac{k\mu}{J^2} \right] + \left[ u - \frac{k\mu}{J^2} \right] = 0$$

$$\Rightarrow \frac{d^2 U}{d\theta^2} + U = 0 \quad \text{where } U = \underline{u - \frac{k\mu}{J^2}}$$

$$\Rightarrow U = A \cos(\theta - \theta_0)$$

$$u - \frac{k\mu}{J^2} = A \cos(\theta - \theta_0)$$

$$\Rightarrow \underline{u = \frac{k\mu}{J^2} + A \cos(\theta - \theta_0)}$$

$$= \frac{k\mu}{J^2} \left[ 1 + \frac{AJ^2}{k\mu} \cos(\theta - \theta_0) \right]$$

But what is A??

$$u + \frac{d^2u}{d\theta^2} = \frac{-\mu F(u)}{J^2 u^2}$$

Q. A particle of mass is moving under spiral motion

(a)  $r = e^\theta$ . Find Force

$$u = e^{-\theta}$$

$$e^{-\theta} + e^{-\theta} = \frac{-\mu F(u)}{J^2 u^2}$$

$$\Rightarrow -2u^3 \frac{J^2}{\mu} = F(u)$$

[eliminate  $\theta$ ]

$$\Rightarrow F(u) = -\frac{2J^2}{\mu} u^3$$

$$F \propto \left(\frac{1}{r^3}\right)$$

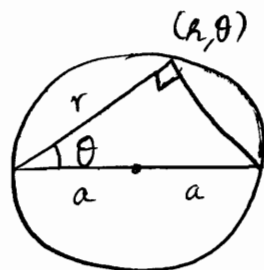
Q. (b)  $r = 2A \cos \theta$

$$u = \frac{1}{2A \cos \theta}$$

$$\frac{1}{2A \cos \theta} + \frac{d}{d\theta} \left[ \frac{1}{2A} \cdot \frac{(+\sin \theta)}{\cos^2 \theta} \right]$$

$$= \frac{1}{2A \cos \theta} + \frac{1}{2A} \frac{d}{d\theta} \left[ \frac{\sin \theta}{\cos^2 \theta} \right]$$

$$\Rightarrow F(u) = \frac{-J^2}{\mu} \left[ u^3 + 8A^2 u^5 - u^3 \right] \left[ \frac{1}{r^5} \right]$$



1) All planets move around sun in <sup>closed</sup> elliptical orbits, sun being at 1 of the focus of the ellipse.

2) All planets moving in elliptical orbits sweep out equal areas in equal time, i.e. Areal velocity is const.

$$\frac{dA}{dt} = \text{const.}$$

3) Time Period squared is proportional to  $a^3$

ie.  $T = \frac{2\pi}{\sqrt{GM}} a^{3/2}$

Proof of 2

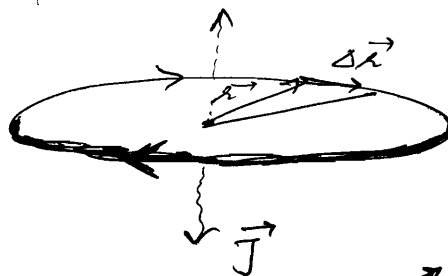
$$\mu r^2 \dot{\theta} = \text{const.}$$

$$\Delta \vec{A} = \frac{1}{2} \vec{r} \times \Delta \vec{r}$$

$$\frac{\Delta \vec{A}}{\Delta t} = \frac{1}{2} \frac{\vec{r} \times \Delta \vec{r}}{\Delta t} = \frac{1}{2} \vec{r} \times \frac{\Delta \vec{r}}{\Delta t}$$

$$\lim_{\Delta t \rightarrow 0} \left( \frac{d\vec{A}}{dt} \right) = \frac{1}{2} \vec{r} \times \left( \frac{d\vec{r}}{dt} \right) = \frac{1}{2} \vec{r} \times \mu \vec{v}$$

$$\Rightarrow \boxed{\frac{d\vec{A}}{dt} = \frac{\vec{J}}{2\mu}}$$



→  $\Delta t$  is a scalar, so can be attached to any ~~factor~~.

$$\Rightarrow \left( \frac{d^2 \vec{x}}{dt^2} \right) = - \frac{J^2 u^2}{\mu^2} \left[ \frac{d^2 u}{d\theta^2} \right]$$

do not forget this  $\mu$

$$\Rightarrow \mu \left[ - \frac{J^2 u^2}{\mu^2} \left[ \frac{d^2 u}{d\theta^2} \right] - \frac{1}{u} \frac{J^2 u^4}{\mu^2} \right] = F(u)$$

$$\Rightarrow - \frac{J^2 u^2}{\mu} \left( \left[ \frac{d^2 u}{d\theta^2} \right] + u \right) = F(u)$$

$$\Rightarrow \boxed{\frac{d^2 u}{d\theta^2} + u = \frac{-\mu F(u)}{J^2 u^2}}$$

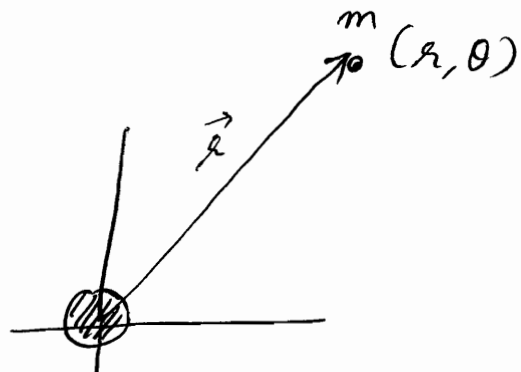
differential equation of central orbit

## Planetary Motion

Motion by virtue of mass

$$\vec{F} = - \frac{GMm}{r^2} \hat{r}$$

$$F(u) = -GMm u^2$$



$$\Rightarrow J = \mu r^2 \dot{\theta} = \text{const}$$

$$= \mu (r^2 \dot{\theta}) \hat{n} \quad (\star)$$

Trajectory of Motion will be given by soln of

$$\underline{F(r) = \mu [ \ddot{r} - r \dot{\theta}^2 ]}$$

$$\frac{1}{r} = \frac{1 + e \cos(\theta - \theta_0)}{l} = \text{~~some~~ } f(\theta)$$

Always remember!!

We have to eliminate time to get  $\searrow$  trajectory.

$$\boxed{\left(\frac{d\theta}{dt}\right) = \left[\frac{J}{\mu r^2}\right]}$$

$$\underline{\frac{dr}{dt}} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = \underline{\frac{J}{\mu r^2}} \left(\frac{dr}{d\theta}\right)$$

Also choose  $\frac{1}{r}$  as  $u$  in order to get

$$u = f(\theta)$$

$$\underline{u = \frac{1}{r}}$$

$$\frac{dr}{dt} = \frac{J}{\mu} u^2 \left(\frac{du}{d\theta}\right) \cdot \left(\frac{dr}{du}\right)$$

$$\boxed{\left(\frac{dr}{dt}\right) = -\frac{J}{\mu} \left(\frac{du}{d\theta}\right)}$$

$$\frac{dr}{du} = \frac{-1}{u^2}$$



by using reduced mass.

$$M\ddot{\vec{r}}_1 = M \frac{d^2\vec{r}_1}{dt^2} = F_{1,2} \Rightarrow \frac{d^2\vec{r}_1}{dt^2} = \left(\frac{F}{M}\right)$$

Also  $\frac{d^2\vec{r}_2}{dt^2} = +\left(\frac{\vec{F}_{3,1}}{m}\right) = -\left(\frac{\vec{F}_{1,2}}{m}\right)$

$$\frac{d^2}{dt^2} (\vec{r}_1 - \vec{r}_2) = \vec{F}_{1,2} \left[ \frac{m_1 + m_2}{m_1 m_2} \right]$$

$$\boxed{\left(\frac{mM}{M+m}\right) \frac{d^2}{dt^2} (\vec{r}_{12}) = \vec{F}_{1,2}}$$

$$\Rightarrow \mu \ddot{\vec{r}} = \vec{F}$$

$$\mu \ddot{\vec{r}} = F(r)$$

$$\mu = \frac{mM}{m+M}$$

$$= \frac{mM}{1 + \frac{m}{M}}$$

$$M \gg m$$

$$\approx mM$$

for easy of solving,  
we ~~can~~ use  $\Lambda$  in  
place of  $\mu$ , which  
can be confused with  $\mu$

$$\mu a_r = F(r)$$

$$\mu a_\theta = 0$$

$$\Rightarrow \frac{\mu}{r} \frac{d}{dt} [r^2 \dot{\theta}] = 0$$

← hence its equivalent to  
observer sitting at mass  $M$   
and seeing a body of mass  
 $\mu = \frac{Mm}{M+m}$  performing a motion  
about it...

Note that using reduced mass is  
equivalent to observing motion from  
 $M$  by applying Pseudo Force upon  $m$ .

$$\vec{a} = [\ddot{r} - r\dot{\theta}^2] \hat{r} + \frac{1}{r} \frac{d}{dt} [r^2 \dot{\theta}] \hat{\theta}$$

For Central Force Motion,

$$\vec{F} = F(r) \hat{r}$$

◦ Note that 'k' is POSITIVE.

eg.  $F(r) = -\frac{k}{r^2}$

if  $k = +GMm \Rightarrow$  Gravitational Force

$F(r) = \frac{k}{r^2}$ ,  $k = +\frac{1}{4\pi\epsilon_0} q_1 q_2 \Rightarrow$  Electrical Force

$F(r) = r^n$  (in general)

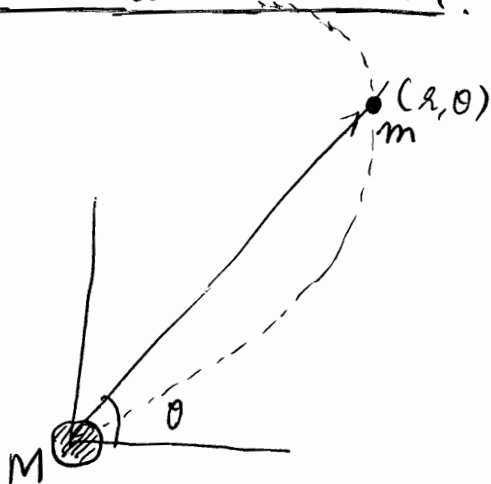
ध्यान दें कि  $k$  में कीस sign देती है, in the expression of force

also  $\vec{F} = 0 \hat{\theta}$  [no component in direction  $\perp$  to radius]

If  $m$  moving around  $M$ ,

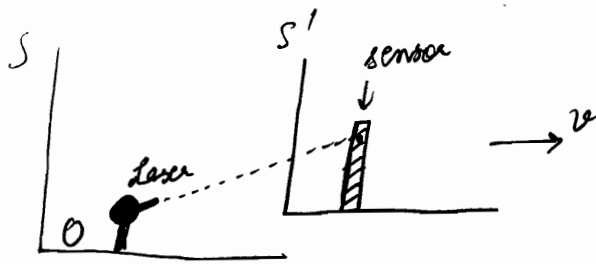
If trajectory is known, only 1 coordinate is sufficient to know the locus.

We usually have frame of reference at centre of 'M' which coincides with COM.



$$\frac{l}{r} = 1 + e \cos \theta$$

# Doppler Effect



do this derivation!!

⊙ Whenever laser is fired, a small light is blinked on sensor which makes noise.

Let 2 lasers be fired at  $t=0$  and  $t=T$

and let observation be made at sensor at  $t=0$  and  $t=t'$

The observer O will observe sensor at  $t = \alpha t'$

He will think that extra time taken is due to covering of extra distance  $\Delta x$

$$\Delta x = \alpha (\Delta x' + v \Delta t') = \alpha v \Delta t' = \alpha v t'$$

Hence for observer O,

$$\Delta t' = T + \left( \frac{\alpha v t'}{c} \right)$$

$$\Rightarrow \Delta t' \left[ 1 - \frac{v}{c} \right] = T$$

$$\Rightarrow t' \frac{\sqrt{1 - \frac{v}{c}}}{\sqrt{1 + \frac{v}{c}}} = T$$

$$\Rightarrow \boxed{\gamma' = \gamma \frac{\sqrt{1 - \frac{v}{c}}}{\sqrt{1 + \frac{v}{c}}}}$$

⬅️ दूरे जा रहा है तो frequency कम ही होगी !!

$$\left. \begin{aligned} \approx \gamma' &= \gamma \left( 1 - \frac{v}{2c} \right) \left( 1 - \frac{v}{2c} \right) = \gamma \left( 1 - \frac{v}{c} \right) \\ \Rightarrow \boxed{\frac{\Delta \gamma}{\gamma} &= \pm \left( \frac{v}{c} \right)} \end{aligned} \right\} \begin{array}{l} \text{used in} \\ \text{Mössbauer} \\ \text{effect} \end{array}$$

⊙ extending the same logic to transverse

$\Delta x$

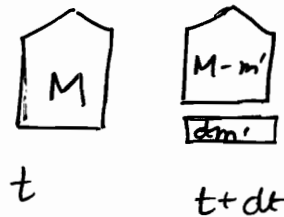
being removed in form of gases.

$$\text{i.e. } M = M_{\text{initial}} - m'$$

$$\text{i.e. } dM = -dm' \quad \text{--- (1)}$$

let at any instant  $t$ , mass be  $M$  and in differential time  $dt$ ,  $dm'$  be removed.

Applying Equation of Motion



$$F_{\text{ext}} \cdot dt = d\vec{p}$$

$$= p_f - p_i$$

$$= \left[ (M - dm') (\vec{v} + d\vec{v}) + \cancel{dm'} (\vec{u}_{\text{exh}} + \vec{v} + d\vec{v}) \right] - [M\vec{v}]$$

$$\Rightarrow F_{\text{ext}} \cdot dt = M d\vec{v} + \vec{u}_{\text{exh}} dm'$$

$$F_{\text{ext}} = -Mg$$

Resolving components along <sup>direction</sup> of motion

$$\Rightarrow -Mg dt = M dv - u_{\text{exh}} dm'$$

$$\Rightarrow M dv = u_{\text{exh}} dm' - Mg dt$$

$$\Rightarrow dv = u_{\text{exh}} \frac{dm'}{M} - g dt$$

Integrating & using (1)

$$\Rightarrow \int_{v_0}^{v_f} dv = -u_{\text{exh}} \int_{M_0}^M \frac{1}{M} dM - \int_0^t g dt$$

⇒

$$v_f - v_o = v_{esh} \ln \left( \frac{M_o}{M} \right) - gt$$

### Greek Alphabets

24 alphabets

|   |   |         |   |   |         |
|---|---|---------|---|---|---------|
| A | α | Alpha   | Τ | τ | Tau     |
| B | β | Beta    | Υ | υ | Upsilon |
| Γ | γ | Gamma   | Φ | φ | Phi     |
| Δ | δ | Delta   | Χ | χ | Chi     |
| E | ε | Epsilon | Ψ | ψ | Psi     |
| Z | ζ | Zeta    | Ω | ω | Omega   |
| H | η | Eta     |   |   |         |
| Θ | θ | Theta   |   |   |         |
| I | ι | Iota    |   |   |         |
| K | κ | Kappa   |   |   |         |
| Λ | λ | Lambda  |   |   |         |
| M | μ | Mu      |   |   |         |
| N | ν | Nu      |   |   |         |
| Ξ | ξ | Xi      |   |   |         |
| Ο | ο | Omicron |   |   |         |
| Π | π | Pi      |   |   |         |
| Ρ | ρ | Rho     |   |   |         |
| Σ | σ | Sigma   |   |   |         |

Underdamped :  $Ae^{-\alpha t} \sin(\omega_d t + \phi_0)$

Critical :  $(A+Bt) e^{-ct}$

Overdamped :  $e^{-ct} [Ae^{-\omega_1 t} + Be^{\omega_2 t}]$

are general and not like starting from maxima & then decreasing. Depending on  $\phi_0, A, B$  we can start from any point.

### Components of angular velocity in terms of Euler angles

Angular velocities are defined in terms of  $(\dot{\phi}, \dot{\theta}, \dot{\psi})$  and  $(\phi, \theta, \psi)$

$$\dot{\phi} = \dot{\phi} \hat{k}_1 = \dot{\phi} \hat{k}$$

$$\dot{\theta} = \dot{\theta} \hat{i}_2 = \dot{\theta} \hat{i}_1$$

$$\dot{\psi} = \dot{\psi} \hat{k}' = \dot{\psi} \hat{k}_2$$

We can express  $\hat{i}_1, \hat{k}_2$  in terms of  $(\phi, \theta, \psi)$  and  $\hat{i}, \hat{j}, \hat{k}$  via matrix relations or in terms of  $(\phi, \theta, \psi)$  and  $\hat{i}', \hat{j}', \hat{k}'$  via inverse matrices. Here, taking (1) inverse is easy (simply transpose).

$$\vec{\omega} = \dot{\phi}' \hat{i}' + \dot{\theta}' \hat{j}' + \dot{\psi}' \hat{k}'$$

