RISHAV GUPTA
9873318559

ELECTROMAGNETISM

PHYSICS - 4

(★) $\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(\vec{r})$

(★) Avogadro's Number में Prime ही Prime हैं, तभी साला याद नहीं होता : $\underline{\underline{N_0 = 6.023 \times 10^{23}}}$

(★) $\zeta(4) = \left(\frac{\pi^4}{90}\right)$

(Stefen Boltzmann)

$\zeta\left(\frac{3}{2}\right) = 2.6$

(Bose Einstein Condensate)

$$\zeta(x+1)\,\Gamma(x+1) = \int_0^\infty \frac{x^n}{e^x - 1}\,dx$$

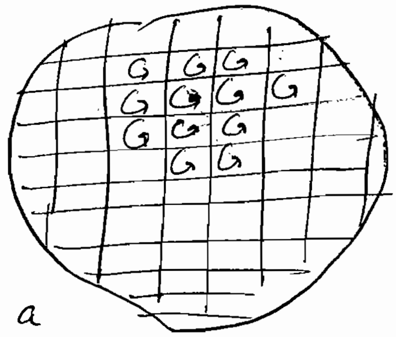
By defn

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

eg. $\zeta(4) = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots$

Magnetic Shell

✓ Magnetic Shell is a theoretical concept which can be regarded as the cause of magnetic field

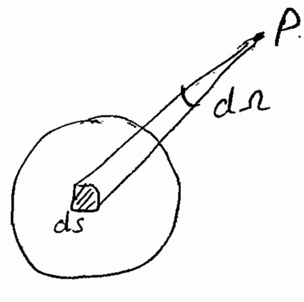


Magnetic Shell is a thin sheet of magnetic material magnetized in such a way that magnetization is perpendicular to the surface of sheet everywhere.

It may be regarded as a large number of very short magnetic dipoles placed edge to edge with similar dipoles pointing in same direction.

Magnetic potential due to magnetic shell at any point P subtending solid angle $d\Omega$ is

$$\phi = \frac{I}{4\pi} d\Omega = \frac{I}{4\pi}\left(\frac{\vec{ds}\cdot\vec{r}}{r^3}\right)$$



If the shell is divided into small elements of small area $\vec{ds}$, we say that each element corresponds to magnetic ~~moment~~ moment

$$\vec{m} = I\,\vec{ds} \Rightarrow I = \frac{|\vec{m}|}{|\vec{ds}|}$$

This is also called strength of the cell i.e. magnetic moment per unit area.

ELECTRICITY & MAGNETISM (1)

* 4Q : 60×4 = 240

If all correct : 0·7×240 = 168

Total : 336

$\boxed{\text{16 lectures}}$

8 + 6 + 2

statics dynamics current

* 15 classes course

* 3 units :

Q6 ① Electrostatics & Magnetostatics ⑥

② Current Electricity ③

③ EM Theory & Blackbody Radiation ⑥ - ⑦

Q7 ← Most scoring question

(a) Electrostatics & Magnetostatics Electrodynamics

(i) Field & Potential due to Dipole, Dipole-Dipole interactions, Multipole expansion of Potential,

(ii) Laplace & Poisson Equation & simple applications

(iii) Method of electrical images

(iv) Dielectric & Polarization

(v) Boundary Value Problem of Conducting & Dielectric Sphere in Uniform Field

(vi) Magnetostatics : Magnetized Sphere in Uniform Field
Ferromagnetic Material & Hysteresis

Prerequisites

Field
Potential
Energy

Electrostatics : stationary charges

Electro Magnetism : moving charge

Field & Potential

$$\vec{F} = -\vec{\nabla} U$$

Potential Energy

$\circledast$ $\vec{\nabla} U \cdot \vec{dr} = dU$

$$\boxed{\Delta U = -\int \vec{F} \cdot \vec{dr}}$$

Change in Potential Energy

dividing by unit mass (or charge)

$\vec{E} = -\vec{\nabla} V$ ← Potential   where $\vec{E} = \dfrac{\vec{F}}{m}$ & $V = \dfrac{U}{m}$

or

$\vec{E} = \dfrac{\vec{F}}{q}$ & $V = \dfrac{U}{q}$

$$\Rightarrow \boxed{\Delta V = -\int E \cdot dr}$$

change in Potential

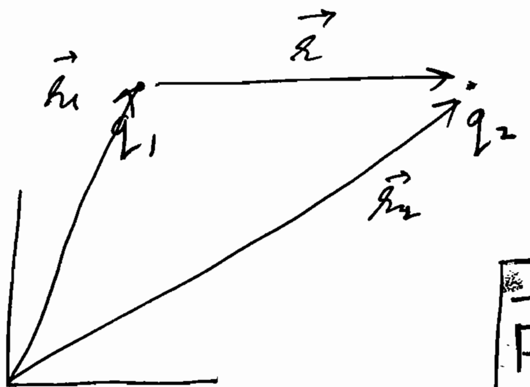
if $\vec{F}_{known} \Rightarrow \vec{E}_{known} \Rightarrow dV_{known} \Rightarrow dU_{known}$

$\boxed{\epsilon : \text{Permittivity}}$

Coulumb's law

$$\vec{F} = \left(\frac{1}{4\pi\varepsilon}\right)\frac{q_1 q_2}{r^2}\,\hat{r}$$

$\boxed{\varepsilon = \varepsilon_0 k}$



$\vec{r} = \vec{r}_2 - \vec{r}_1$

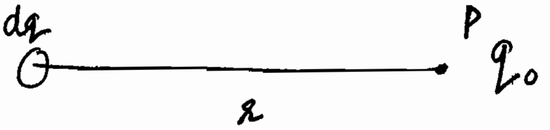
k : dielectric const.
or
relative permittivity

$$\boxed{\vec{F}_{medium} = \left(\frac{\vec{F}_{air}}{k}\right)}$$

$k \geq 1$

$\Rightarrow E_{medium}$ reduced $\Rightarrow \Delta V_{medium}$ reduced

Let us consider a system :-



$$d\vec{F} = \frac{1}{4\pi\varepsilon_0 k} \frac{dq\, q_0}{r^2} \cdot \hat{r}$$ (from Coulumb's law)

experimental law

empirical observations

$$d\vec{E} = \frac{dF}{q_0} = \text{Field generated by } dq$$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0 k} \left(\frac{dq}{r^2}\right) \hat{r}$$

$$\boxed{\vec{E} = \int d\vec{E} = \frac{1}{4\pi\varepsilon_0 k} \int \frac{dq}{r^2} \hat{r}}$$

This is how Coulumb law is used to find out $\vec{E}$ at a point $\vec{r}$.

$$V = -\int \vec{E} \cdot d\vec{r}$$

$$U = qV$$

$$dq = \begin{cases} \lambda\, dl & \text{(1-dimension)} \\ \sigma\, da & \text{(2-dimension)} \\ \rho\, dv & \text{(3- dimension)} \end{cases}$$

Also note $$\boxed{\frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3} = -\vec{\nabla}\left(\frac{1}{r}\right)}$$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0 k} \frac{dq}{r^2} \hat{r} = -\vec{\nabla}\left(\frac{1}{4\pi\varepsilon_0 k} \frac{dq}{r}\right)$$

Now $d\vec{E} = -\vec{\nabla} V$

$$\Rightarrow \phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{dq}{r}\right)$$

→ ϕ is Potential provided it is Work Done per unit charge.

$$dW = \vec{F}\cdot\vec{dr} = q_0\vec{E}\cdot dr$$

$$\vec{E}\cdot\vec{dr} = \frac{dW}{q} = -\vec{\nabla}\phi\cdot\vec{dr}$$

$$= -\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(dx\,i + dy\,j + dz\,k)$$

$$= -\,d\phi$$

$$\Rightarrow d\phi = -\vec{E}\cdot\vec{dr} = \left(\frac{dW}{q_0}\right) = dV$$

Hence ϕ is Potential

$$\Rightarrow dV = \frac{1}{4\pi\varepsilon_0}\left(\frac{dq}{r}\right)$$

V_{ref} at $V_\infty = 0$ [Convention]

$$\int_0^V dV = \int \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = -\int_\infty^r E\cdot dr$$

Gauss law

Flux of Electric Field through a closed surface is $\left(\frac{1}{\varepsilon_0}\right)$ times charge enclosed in the surface.

$$\boxed{\text{Flux} = \oint_S \vec{E} \cdot d\vec{s} = \frac{1}{\varepsilon_0} (q_{enc})}$$

q_{enc} is q_{total} i.e. $q_{free} + q_{bound}$

From Fundamental law of divergence

$$\oint \vec{E} \cdot d\vec{s} = \int \vec{\nabla} \cdot \vec{E} \, dv = \frac{1}{\varepsilon_0} \int \rho \, dv$$

$$\Rightarrow \int \left(\vec{\nabla} \cdot \vec{E} - \frac{\rho}{\varepsilon_0} \right) dv = 0$$

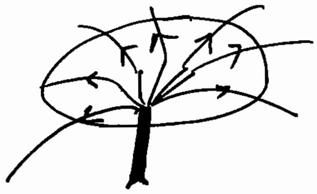
$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}}$$

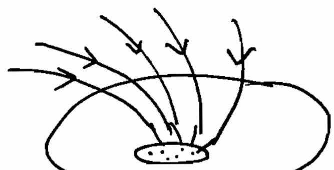
[Note that the Gauss law is independent of medium]

For source : divergence is ⊕ve ✪ दान देना is a ⊕ve thing !!

For sink : divergence is ⊖ve

Positive divergence





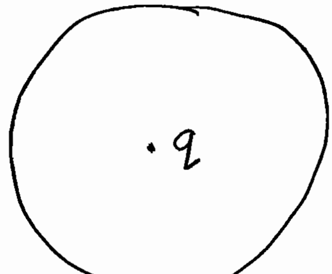
Negative divergence

→ The calculation of Gauss law is simpler when surface chosen is symmetric about E s.t. $\oint E \cdot ds$ can be calculated easily. Such a closed symmetric surface is called GAUSSIAN SURFACE.

⊛ Note that in gravitation multiplier was $(-4\pi G)$ while here the multiplier is $\left(\frac{+1}{\varepsilon_0}\right)$

Proof

Consider a symmetric Gaussian surface around point charge q.



Flux of $\vec{E} = \oint_s \vec{E} \cdot \vec{ds}$

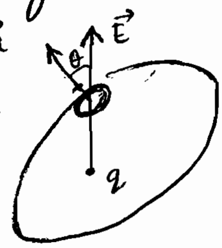
$$= \oint \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \cdot \vec{ds}$$

$$= \frac{1}{4\pi\varepsilon_0} q \frac{1}{r^2} \cdot 4\pi r^2$$

$$= \left(\frac{q}{\varepsilon_0}\right)$$

General Proof of Gauss law

Flux due to q at a differential element da of the gaussian surface



$$d\phi = \vec{E} \cdot \vec{da}$$

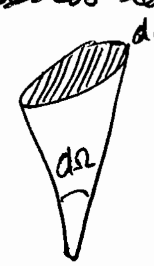
$$= E\,da \cos\theta$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} da \cos\theta$$

~~from coulomb's law~~

$$= \frac{q}{4\pi\varepsilon_0} \cdot \left(\frac{da \cos\theta}{r^2}\right)$$

$$= \frac{q}{4\pi\varepsilon_0} \cdot d\Omega$$



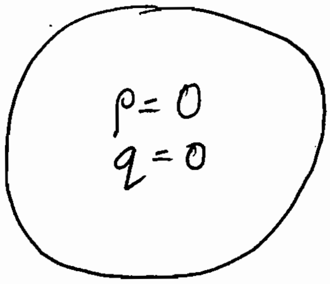
$$\Rightarrow \phi = \int d\phi = \frac{q}{4\pi\varepsilon_0} \cdot 4\pi = \left(\frac{q}{\varepsilon_0}\right)$$

⊙ Also Ω subtended at an external point = 0
⇒ No contribution of external charges

⊙ There are 5 methods to find $\vec{E}$ (that means all 4 variables of Electrostatics :-

1. Coulumb's law
2. Gauss law
3. Poisson & Laplace Equation
4. Method Of Electrical Images
5. Graphical Method (not in course)

Conductor



For Conducting Surface: all charge resides on surface

$\Rightarrow$ no charge inside

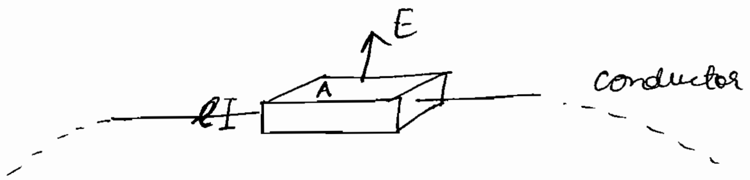
$\Rightarrow$ $\vec{E} = 0$ inside Conductor

$\Rightarrow$ $V = $ Const. Hence whole conductor is equipotential

Potential$_{(inside)}$ = Potential$_{(surface)}$

$$dW = q\,dV = 0$$

$$\vec{E} = \left(\frac{\sigma}{\epsilon_0}\right)\hat{n}$$



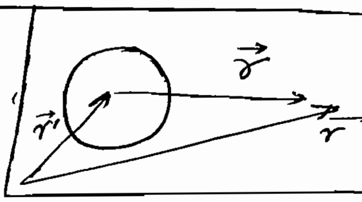
Using Gauss law,

$$E.A = \frac{\sigma.A}{\epsilon_0}$$

$$\Rightarrow E = \left(\frac{\sigma}{\epsilon_0}\right) \qquad \boxed{\vec{E} = \left(\frac{\sigma}{\epsilon_0}\right)\hat{n}}$$

General Proof of Gauss law:

We know, $E(r) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \rho(r')\, d\tau'$



where $\vec{\mathscr{r}} = (\vec{r} - \vec{r}')$

P-87 Griffith

$$\Rightarrow \nabla\cdot E = \frac{1}{4\pi\epsilon_0} \int \nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) \rho(r')\, d\tau'$$

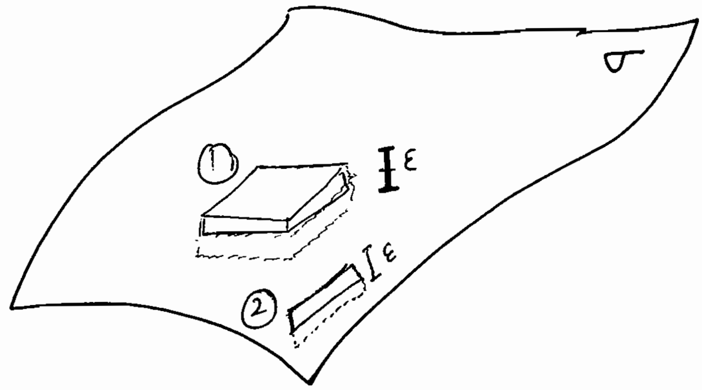
$\boxed{\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(r)}$ Correct

Now $\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\mathscr{r})$

$$\Rightarrow \nabla\cdot E = \frac{1}{4\pi\epsilon_0}\int 4\pi\, \delta^3(\mathscr{r})\, \rho(r')\, d\tau' = \frac{1}{\epsilon_0}\int \delta^3(r - r')\rho(r')\, d\tau'$$

$$= \frac{1}{\epsilon_0}\rho(r) \quad : \text{Gauss law}$$

(*) Consider a charged boundary of surface charge σ.



Consider a match box as gaussian surface which is wafer thin, extending barely over the edge in each direction

From Gauss law,

$$\oint_S \vec{E}.\vec{da} = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

(if σ varies from point to point, choose a small A)

Now sides contribute nothing to flux as $\epsilon \to 0$

$$\Rightarrow \boxed{E^{\perp}_{above} - E^{\perp}_{below} = \left(\frac{\sigma}{\varepsilon_0}\right)} \quad \text{—— (1)}$$

$\Rightarrow$ The normal component of $\vec{E}$ is discontinuous by an amount $\left(\frac{\sigma}{\varepsilon_0}\right)$ at any boundary. In particular, $E^{\perp}$ is continuous where there is no surface charge, eg. at the surface of a uniformly charged solid sphere.

The tangential component of $\vec{E}$, in contrast, is always continuous. Since $\vec{E}$ is conservative,

We know $\oint E \cdot dl = 0$

Applying this to thin rectangular loop, where $\epsilon \to 0$

$$\boxed{E^{\parallel}_{above} = E^{\parallel}_{below}} \quad \text{—— (2)}$$

Combining (1) and (2),

$$\boxed{\vec{E}_{above} - \vec{E}_{below} = \left(\frac{\sigma}{\varepsilon_0}\right)\hat{n}}$$

Meanwhile, the potential is continuous across any boundary since

$$V_{above} - V_{below} = -\int_a^b \vec{E}\cdot\vec{dl}$$

as the path shrinks to zero, so does the integral,

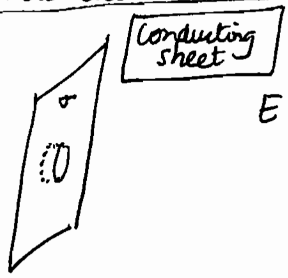
$$\boxed{V_{above} = V_{below}}$$

---

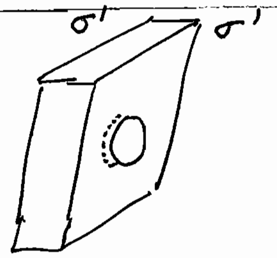
(1) $\vec{\nabla}\cdot\vec{E} = \left(\frac{\rho_{enc}}{\varepsilon_0}\right)$: general gauss law. Valid even for dielectrics.

If you want concomitance with Columb's law $\vec{E} = \frac{1}{4\pi\varepsilon_0 k}\cdot\frac{q}{r^2}\hat{r}$ we need to consider $\rho_b = -\vec{\nabla}\cdot\vec{P}$. Then there will be no contradition between the two.

---

(2) 

$E = \frac{\sigma}{2\varepsilon_0}\hat{r}$



$E = \left(\frac{\sigma'}{\varepsilon_0}\right)$

But $\sigma' = \frac{\sigma}{2}$

$\Rightarrow E = \left(\frac{\sigma}{2\varepsilon_0}\right)\hat{r}$

How so ever thin the sheet is, we may consider it as a case of part II

Hence no contradiction.

EM (2)

14/01/2012

Blackbody Radiation

1850 - 1900 : Classical Study

after 1900 : Quantum Study

Thermal Radiation typically lies in infrared region. As the body is heated more, the spectrum changes from infrared to blue light (more energy). (Concomitant with vibrational energy)

Blackbody Radiation can be regarded as either Thermodynamic System or Electromagnetic System (before 1900). Now we treat it as Quantum System.

Observed Fact that $\nu \propto T$ or $\lambda \propto \frac{1}{T}$

by observing colour of light and temperature of Blackbody.

Blackbody means any body which is at higher Temperature than surrounding temperature. Mode of heat transfer is radiation. Note that every body can be blackbody while "Perfect Blackbody" is an ideal concept.

Wein Observed the blackbody radiation and observed

$$\lambda \propto \frac{1}{T} \quad \text{i.e.} \quad \lambda T = \text{const}$$ Weins displacement law

He considered it thermodynamically and gave Wein's distribution law. (System filled with adiabatically expanding assembly of particles)

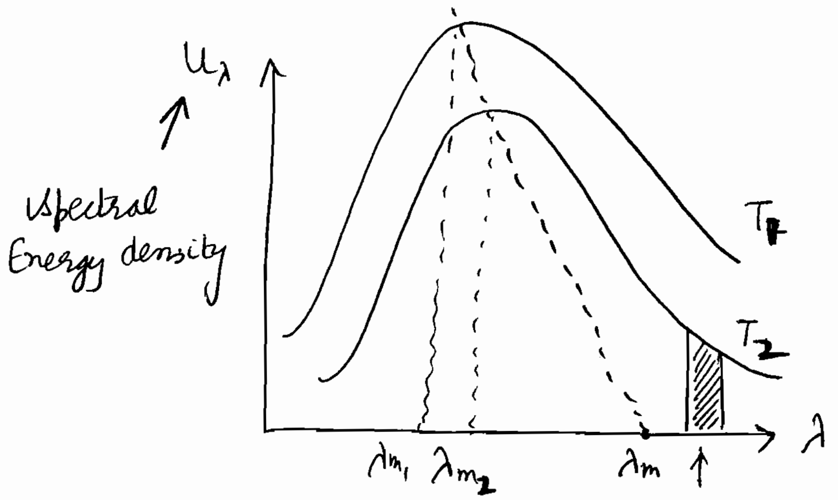
Rayleigh - Jean Observed it as an electromagnetic phenomenon.

Stephen Boltzmann came even before Wein's law. It says

Energy radiated by blackbody $[u \propto T^4]$ T = Temperature of Blackbody

[redacted]

"Lummer & Pringshien" heated various bodies and studied u_λ vs λ. Empirically they found:



$T_1 > T_2$

$\Leftarrow u = \int u_\lambda \, d\lambda$

Area = $u_\lambda \, d\lambda$

= energy emitted between λ and $(\lambda + d\lambda)$

Wein $\leftarrow$ lower wavelength side or higher frequency side

$\rightarrow$ Rayleigh Jean: higher wavelength side or lower frequency side

u_λ [redacted] : energy density

u : energy $= \int u_\lambda \, d\lambda$

Stephen Boltzmann's Formula

Area$_1 \propto T_1^4$

Area$_2 \propto T_2^4$

$T_1 > T_2$

$\lambda_{m_1} < \lambda_{m_2}$

$\lambda_{m_1} \propto \frac{1}{T}$

Wein's Formula

$\rightarrow$ From these empirical observations, all the 3 laws could be observed:

1. Stephen Boltzmann law
2. Wein's law
3. Rayleigh Jean law

- Rayleigh Jean's law : $u_\lambda \, d\lambda = \left(\frac{8\pi}{\lambda^4} \, d\lambda \right) kT$

We can also write $u_\nu \, d\nu = \underbrace{\left(\frac{8\pi \nu^2}{c^3} \, d\nu \right)}_{\text{Standing Waves' modes of vibration}} \underbrace{kT}_{\text{energy per mode}}$

- Wein's distribution law : $u_\lambda \, d\lambda = \frac{A}{\lambda^5} \, f(\lambda T) \, d\lambda$

- Stephen Boltzmann's law : $\int u_\lambda \, d\lambda \propto T^4$

Both Rayleigh and Wein were not able to explain the curve. Both believed that E is continuous.
Till now, no concept of Photon.

Now came Plank. He gave Planck's Blackbody Radiation Formula. He considered

$u_\lambda \, d\lambda = \left(\begin{array}{l} \text{No. of modes of vibration in wavelength region} \\ \lambda \text{ and } (\lambda + d\lambda) \text{ in unit volume} \end{array} \right) \times (\text{Average Energy per mode})$

Now we know,

$$g(\nu)\, d\nu = \frac{d^3\nu \, d^3p}{h^3} = \frac{V}{h^3} \cdot 4\pi p^2 \, dp = V \cdot \frac{4\pi \nu^2}{c^3} \, d\nu$$

For 2 modes of Polarization $= V \cdot \frac{8\pi}{\lambda^4} \, d\lambda$

Plank took the formula from Rayleight Jean for no. of modes and consider Avg. energy per mode = $\boxed{\frac{\left(\frac{hc}{\lambda}\right)}{\left(e^{hc/\lambda kT} - 1\right)}}$

Planck's Radiation Formula

$\Rightarrow$ $\boxed{u_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{d\lambda}{e^{(hc/\lambda kT)} - 1}}$ Planks Blackbody Distribution Formula

We can derive easily from [redacted] distribution
Maxwell Boltzmann

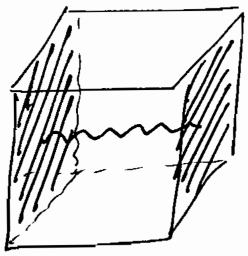
$$u = g(\nu)\, d\nu \; f(\epsilon)\, d\epsilon$$

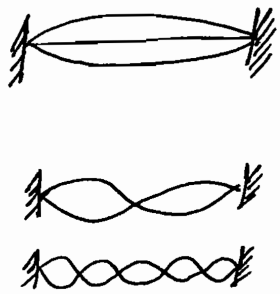
$$\langle \epsilon \rangle = \frac{h\nu}{e^{h\nu/kT} - 1}$$

at $T \uparrow$: $\langle \epsilon \rangle \rightarrow kT$

Hence classical blackbody radiation explained.

<u>To find no. of Modes</u> :





different modes of standing waves.

$$L_x = n_x \cdot \left(\frac{\lambda}{2}\right) = \left(\frac{n_x c}{2\nu}\right) \qquad \nu = \left(\frac{n_x c}{2L_x}\right)$$

$$L_y = n_y \left(\frac{\lambda}{2}\right)$$

$$L_z = n_z \left(\frac{\lambda}{2}\right)$$

Let
$$L_x = L\cos\alpha = n_x \frac{\lambda}{2}$$
$$L_y = L\cos\beta = n_y \frac{\lambda}{2}$$
$$L_z = L\cos\gamma = n_z \frac{\lambda}{2}$$

(What is L....
इसपे दिमाग खराब
मत करो)

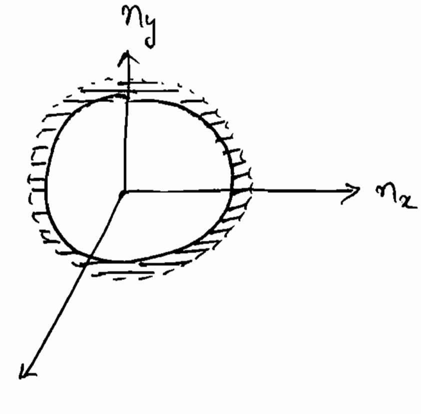
We know, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\frac{(n_x^2+n_y^2+n_z^2)}{L^2}\ \frac{\lambda^2}{4} = 1$$

$$\Rightarrow (n_x^2+n_y^2+n_z^2) = \left(\frac{2L}{\lambda}\right)^2$$

It is the equation of sphere.

$\Rightarrow$



⊛ Shaded region is the no. of modes between λ and $(\lambda+d\lambda)$

$$\left[(n_x+dn_x)^2+(n_y+dn_y)^2+(n_z+dn_z)^2\right] = \left[\frac{2L}{\lambda+d\lambda}\right]^2$$

No. of modes between wavelengths λ and $(\lambda+d\lambda)$ (Positive Quadrant only)

$$= \frac{1}{8}\left[4\pi R^2\, dR\right]$$

$$= \frac{1}{8}\cdot 4\pi\ \frac{4L^2}{\lambda^2}\cdot\frac{2L}{\lambda^2}\ d\lambda \qquad \boxed{R=\frac{2L}{\lambda}}$$

$$= V.\ \frac{4\pi}{\lambda^4}\ d\lambda$$

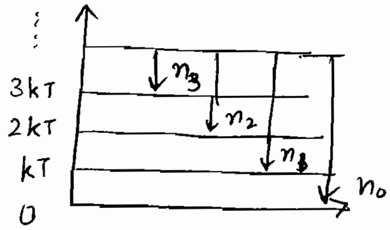
For longitudinal waves : Modes $= V.\ \frac{4\pi}{\lambda^4}\ d\lambda$

For transverse waves : Modes $= V.\ \frac{8\pi}{\lambda^4}\ d\lambda$

[Quantum Results just verifies these results]

Note that

$$\langle \epsilon \rangle = \frac{\sum \epsilon_i n_i}{\sum n_i}$$



Planck had only Maxwell Boltzmann statistics at his disposal.

[Planck proposed $\epsilon = h\nu$ i.e. discrete in nature. It was starting point of Quantum Energy]

$$\boxed{n_i = A e^{-\beta \epsilon_i}} \leftarrow \text{Maxwell distribution}$$

Put $A =$ [redacted] n_0

$$\Rightarrow n_i = A e^{-\beta(h\nu/kT)}$$

$$\Rightarrow n_1 = n_0 e^{-(h\nu/kT)}$$

$$n_2 = n_0 e^{-(2h\nu/kT)}$$

$\vdots$

$$\sum n_i = n_0 + n_0 e^{-(h\nu/kT)} + n_0 e^{-(2h\nu/kT)} + \cdots$$

$$\boxed{N = n_0 \left[\frac{1}{1 - e^{-(h\nu/kT)}}\right]}$$

$$\Rightarrow \langle \epsilon \rangle = \frac{\sum 0 \times n_0 + h\nu n_1 + 2h\nu n_2 + \cdots}{\sum n_i} = \left(\frac{E}{N}\right)$$

$$E = \sum \left[n_0 h\nu e^{-h\nu/kT} + 2 n_0 h\nu e^{-2h\nu/kT} + \cdots \right]$$

$$= n_0 \cdot h\nu e^{-(h\nu/kT)} \left[1 + 2 e^{-h\nu/kT} + 3 e^{-2h\nu/kT} + \cdots \right]$$

Its an AP GP

Let $S = 1 + 2e^{-(h\nu/kT)} + 3e^{-(2h\nu/kT)} + \dots$

$$e^{-h\nu/kT} S = e^{-(h\nu/kT)} + 2e^{-(2h\nu/kT)} + \dots$$

$$S\left[1 - e^{-h\nu/kT}\right] = 1 + e^{-h\nu/kT} + e^{-2h\nu/kT} + \dots$$

$$= \frac{1}{1 - e^{-(h\nu/kT)}}$$

$$\Rightarrow \boxed{E = n_0 h\nu\, e^{-(h\nu/kT)} \cdot \frac{1}{\left[1 - e^{-(h\nu/kT)}\right]^2}}$$

$$\langle \epsilon \rangle = \frac{E}{n_0 \left[\frac{1}{1 - e^{-(h\nu/kT)}}\right]} = \frac{h\nu\, e^{-(h\nu/kT)}}{1 - e^{-(h\nu/kT)}}$$

$$\boxed{\langle \epsilon \rangle = \frac{h\nu}{e^{h\nu/kT} - 1}}$$ ✓

⊛ This is average energy per mode

$$\langle \epsilon \rangle = \epsilon f(\epsilon) = h\nu\left(\frac{1}{e^{\frac{h\nu}{kT}} - 1}\right)$$

Total energy $= g(\epsilon) \cdot \langle \epsilon \rangle \, d\epsilon$

$$\boxed{\begin{aligned} u_\nu \, d\nu &= \frac{8\pi h \nu^3}{c^3} \frac{d\nu}{\left(e^{h\nu/kT} - 1\right)} \\ u_\lambda \, d\lambda &= \frac{8\pi h c}{\lambda^5} \frac{d\lambda}{\left(e^{\frac{hc}{\lambda kT}} - 1\right)} \end{aligned}}$$

Planck's Radiation Formulae

Perfect

We can derive all the 3 classical laws from here :

Stefan Boltzmann Law

$$\int_0^\infty u_\lambda \, d\lambda = U = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3 \, d\nu}{e^{h\nu/kT} - 1}$$

Put $\frac{h\nu}{kT} = x$

$$\nu = x\left(\frac{kT}{h}\right)$$

$$U = \frac{8\pi h}{c^3} \int_0^\infty \left(\frac{kT}{h}\right)^4 \underbrace{\frac{x^3 \, dx}{e^x - 1}} \Rightarrow \left(\frac{\pi^4}{15}\right)$$

$$\int_0^\infty \frac{x^3 \, dx}{e^x - 1} = \underbrace{\left(\frac{\pi^4}{90}\right)}_{\zeta(4)} \times \underbrace{\Gamma(4)}_{\Gamma(4)}$$

$$= \frac{6\pi^4}{90}$$

$$= \left[\frac{8\pi h}{c^3} \cdot \frac{6\pi^4}{90} \cdot \frac{k^4}{h^4}\right] T^4$$

$$\boxed{U = A T^4}$$

U: Energy per unit volume

Hence derived.

S: Energy radiated per unit Area per unit time

$$\Rightarrow S = \frac{1}{4} U.c$$

$\frac{1}{4}$: Correction for falling normally to an area

$$S = \underbrace{\left[\frac{1}{4} \cdot \frac{6 * 8\pi^5 k^4}{90 h^3 c^2}\right]}_{\sigma} T^4 = \sigma T^4$$

$$\boxed{S = \sigma T^4}$$

$$\boxed{k = 1.38 \times 10^{-23}}$$

$\sigma = 5.68 \times 10^{-8}$ SI units

(Stefan Boltzmann's Constant)

<u>Rayleigh Jean's law</u> (बड़ा नाम, बड़ा λ)

$$u_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{e^{hc/\lambda kT} - 1}$$

In higher wavelength region, $e^{\frac{hc}{\lambda kT}} = \left(1 + \frac{hc}{\lambda kT}\right)$

$$\boxed{u_\lambda \, d\lambda = \frac{8\pi \cancel{hc}}{\lambda^5} \cdot \frac{\lambda kT}{\cancel{hc}} d\lambda = \frac{8\pi k T}{\lambda^4} d\lambda}$$

Hence Planck's law converges into Rayleigh-Jean's law at higher wavelength regions.

<u>Wein's Distribution law</u> (छोटा नाम, छोटा λ)

For small λ

$$\boxed{u_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{d\lambda}{e^{(hc/\lambda kT)}} = \frac{A}{\lambda^5} f(\lambda T) \, d\lambda}$$

Hence Planck's law converges into Weins law at lower wavelength regions.

Wein's Displacement Formula

It says $\lambda_m T = \text{Const} = \underbrace{0.2896 \text{ cm K}}$

Wein's Constant (b)

$$\left.\frac{du_\lambda}{d\lambda}\right|_{\lambda_m} = 0 \qquad \left.\frac{d^2 u_\lambda}{d\lambda}\right|_{\lambda_m} < 0$$

$$u_\lambda = 8\pi hc \left[\lambda^{-5} \left(e^{hc/\lambda kT} - 1\right)^{-1} \right]$$

$$-5\lambda^{-6} \left(e^{hc/\lambda kT} - 1\right)^{-1} + \lambda^{-5} - \left(e^{hc/\lambda kT} - 1\right)^{-2} e^{(hc/\lambda kT)} \left(\frac{hc}{kT}\right)\frac{-1}{\lambda^2}$$

$$\Rightarrow \left[-5\lambda + \left(e^{hc/\lambda kT} - 1\right)^{-1} e^{hc/\lambda kT} \left(\frac{hc}{kT}\right) \right] = 0$$

$$\Rightarrow 5 = \left(\frac{hc}{\lambda kT}\right) \frac{e^{hc/\lambda kT}}{e^{hc/\lambda kT} - 1}$$

Let $\frac{hc}{\lambda kT} = x \Rightarrow \boxed{\frac{x e^x}{e^x - 1} = 5}$

Transcedental Equation in x

Solution : $x \simeq 4.95$

$$\frac{hc}{\lambda_m kT} = 4.95 \Rightarrow \boxed{(\lambda_m T)} = \frac{hc}{4.95\, k} = b$$

$$= \boxed{0.2896 \text{ K cm}}$$

(★) **Temperatures of stars are known by studying spectra of stars. Wein's displacement law is used to study it.**

⊙ Proof of $\nabla \times \vec{E} = 0$

Let us prove it for the simplest configuration : a point charge

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

$$\Rightarrow \int_a^b \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0} \cdot \int_a^b \frac{q}{r^2} \cdot dr = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

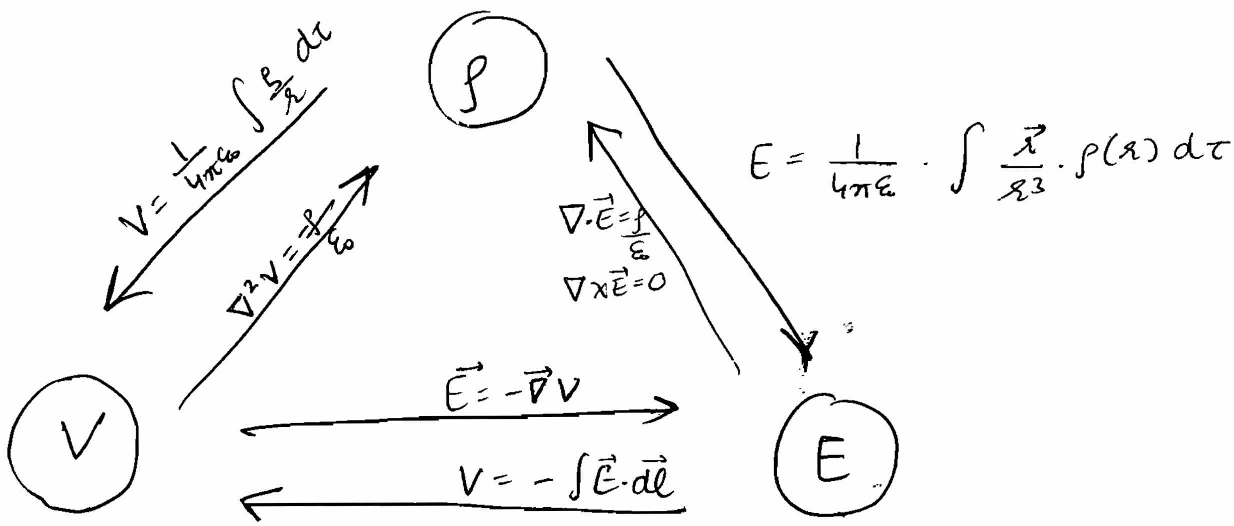
The integral around a closed path i.e. $a = b$

$$= \int_a^a E \cdot dl = \oint E\, dl = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{a}\right] = 0$$

From Stokes theorem

$$\oint E \cdot dl = \int (\vec{\nabla} \times \vec{E}) \cdot d\vec{A}$$

Since R.H.S. $= 0$ for any surface $\Rightarrow \boxed{\vec{\nabla} \times \vec{E} = 0}$

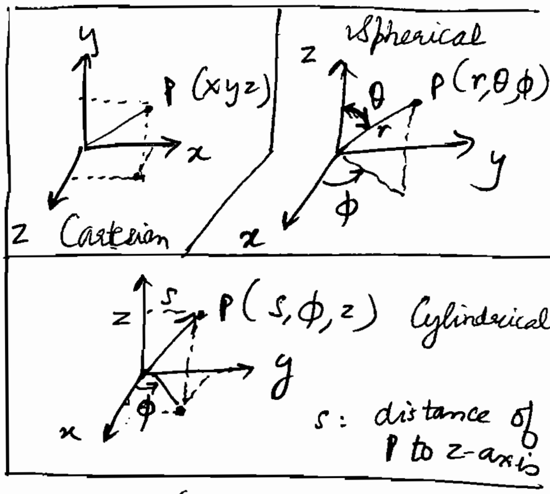


(★) $E_{\text{per unit volume}} = \frac{1}{2}\varepsilon_0 |\vec{E}|^2$
(energy)

This energy is stored in the electric field itself.

E&M (3)

16/01/2012



LAPLACE $ POISSON EQUATION

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \quad : \text{Poisson Eq}^n$$

$$\nabla^2 V = 0 \quad \text{Laplace Eq}^n$$

(Called laplace b'coz it has NOTHING else apart from a LAPLACE in LHS.)

✓ We can apply $[\nabla^2 V]$ either in rectangular (cartesian) coordinates or spherical coordinates.

✓ Note that the 5 methods to calculate electrostatic problems can be used to solve any force-field problems. (G, E, strong, weak)

⊙ Simple applications may include

(i) ρ is given, V is to be determined

eg. ⓐ $\rho = \rho_0 x^2$

★ Note that $V = V(x)$ if $\rho = \rho(x)$ is a correct one!! say if $V = V(x,y)$, then for sure some y term will come in ρ.

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} = -\left(\frac{\rho_0}{\epsilon_0}\right) x^2$$

since $\rho = f(x) \Rightarrow V = f_1(x) \Rightarrow \frac{\partial^2 V}{\partial x^2} = \frac{d^2 V}{dx^2}$

$$\Rightarrow \frac{d^2 V}{dx^2} = -\left(\frac{\rho_0}{\epsilon_0}\right) x^2$$

$$\Rightarrow V = -\left[\frac{\rho_0}{\epsilon_0}\right]\left[\frac{x^4}{12} + Ax + B\right]$$

(b) $\rho = \rho_0 \left(1 - \frac{a^3}{r^3}\right)$, $r \leq a$

1 way : Put $r = \sqrt{x^2+y^2+z^2}$: Its very complicated !!!

2^{nd} way : $\rho = f(r)$

$\Rightarrow V = f_1(r)$

$$\boxed{\nabla^2 \lambda = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \lambda}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \lambda}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 \lambda}{\partial \phi^2}\right)}$$

SPHERICAL COORDINATES

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = -\left(\frac{\rho_0}{\epsilon_0}\right)\left[1 - \frac{a^3}{r^3}\right]$$

$$\Rightarrow \frac{\partial}{\partial r}\left[r^2 \frac{dV}{dr}\right] = -\frac{\rho_0}{\epsilon_0}\left[r^2 - \frac{a^3}{r}\right]$$

$$\Rightarrow r^2 \frac{dV}{dr} = -\frac{\rho_0}{\epsilon_0}\left[\frac{r^3}{3} - a^3 \ln r\right] + A$$

$$\Rightarrow \frac{dV}{dr} = \frac{-\rho}{\epsilon_0}\left[\frac{r}{3} - \frac{a^3 \ln r}{r^2}\right] + \frac{A}{r^2}$$

$$\Rightarrow V = \int \frac{dV}{dr} = \frac{-\rho}{\epsilon_0}\left[\frac{r^2}{6} - \int \frac{a^3}{r^2} \ln r\right] - \frac{A}{r} + B$$

IBP

(ii) V is given, ρ is to be determined

eg. $\underline{\rho = -\epsilon_0 \nabla^2 V}$

11. $V = (x^2+y^2+z^2)$

$\vec{E} = -\vec{\nabla} V = -[2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}]$ Volts/meter

$\rho = -\epsilon_0 \nabla^2 V$

$= -\epsilon_0 (2+2+2) = -6\epsilon_0$ Columb/m^3

* Note that $\vec{E} = -\vec{\nabla}\phi$ is valid only for conservative field. Hence we have to mention, it is conservative as $E = f(x, y, z)$ and not $\hat{x}, \hat{y}, \hat{z}$.

19. $\underline{V = a - b(x^2+y^2) - c\ln(x^2+y^2)}$

Note that it is conservative system

$\Rightarrow E = -\vec{\nabla} V$

$\Rightarrow \nabla\cdot\vec{E} = -\nabla^2 V = \dfrac{\rho}{\epsilon_0}$ $\quad\Rightarrow \rho = -6[-2b + \ldots]$

$\Rightarrow \rho = -\epsilon_0 (\nabla^2 V)$

(iii) Verify ~~the equations~~,

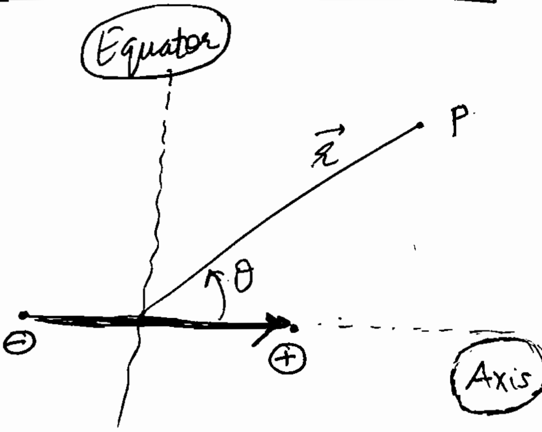
eg. $\underline{V(r,\theta) = \dfrac{1}{4\pi\epsilon_0}\dfrac{p\cos\theta}{r^2}}$

No derivatives involved, hence conservative....

$$\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} =$$

Spherical Coordinates

$$\boxed{\vec{\nabla}\lambda = \left[\frac{\partial \lambda}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \lambda}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial \lambda}{\partial \phi}\hat{\phi}\right]}$$



$$E_r = \frac{1}{4\pi\epsilon_0}\frac{2p\cos\theta}{r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

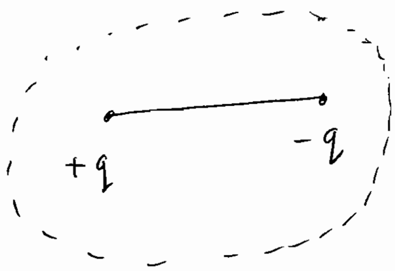
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

Spherical Coordinates

$$\boxed{\vec{\nabla}\cdot\vec{A} = \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right]}$$

Now find $\nabla^2 V = -\frac{\rho}{\epsilon_0}$

$\rho = 0 \Rightarrow \nabla^2 V$ should come out to be 0.



$q_{enc} = 0$

$$\nabla^2 V = \frac{1}{r^2}\left[\frac{-2\cos\theta}{r^2}\right] + \frac{1}{r\sin\theta}\frac{2\sin\theta\cos\theta}{r^3}$$

$$= \frac{[-2\cos\theta + 2\cos\theta]}{r^4} = 0$$

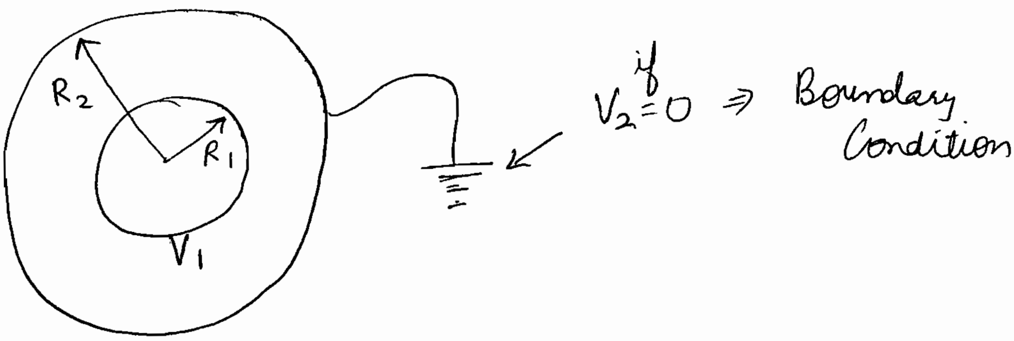
The distance between charges should be of magnitude of Atomic distances. Otherwise it is not a dipole.

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left(2\cos\theta\,\hat{r} + \sin\hat{\theta}\right)$$

$$E(r,\theta) = |\vec{E}| = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\sqrt{3\cos^2\theta + 1}$$

(in languag based problems.

eg. 2 [redacted] concentric spheres, Find out V in between R_1 and R_2.



If non conducting $\rho = \rho_0\left(1 - \frac{a^3}{r^3}\right)$ ✓

If conducting $\rho = 0$ ✓

✓ looking at Symmetric geometory, we can safely conclude

$V = f(r)$

$$\Rightarrow \frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0$$

$$\Rightarrow r^2\frac{dV}{dr} = A$$

$$\Rightarrow \frac{dV}{dr} = \frac{A}{r^2} \quad \Rightarrow V = -\frac{A}{r} + B$$

@ $r = R_1$, $V = V_1$ and @ $r = R_2$, $V = 0$

$$0 = -\frac{A}{R_2} + B \Rightarrow B = \frac{A}{R_2}$$

$$V_1 = \frac{-A}{R_1} + \frac{A}{R_2} \Rightarrow A = \frac{V_1}{\left(\frac{1}{R_2} - \frac{1}{R_1}\right)}$$

Hence = $\boxed{V = \frac{-A}{r} + B}$ .... all known variables.

⊛ do not try to cross verify via Gauss law

$$V_1 = \frac{1}{(4\pi\varepsilon_0)} \frac{Q_{enc}}{R_1}$$

$$\Rightarrow E \cdot 4\pi r^2 = \frac{Q_{enc}}{\varepsilon_0}$$

$$\Rightarrow \int_r^{R_2} dV = -\int_r^{R_2} E \cdot dr$$

THIS IS WRONG AS 1st STEP IS WRONG.

~~Potentials~~ $V = \frac{4\pi\varepsilon_0 Q_{enc}}{r}$ valid for isolated sphere (no other charge like another shell in this case) and assuming E varies similarly from R_1 to ∞ as $\left(\frac{1}{r^2}\right)$

$r_1 < r < r_2$ — 2nd step is correct

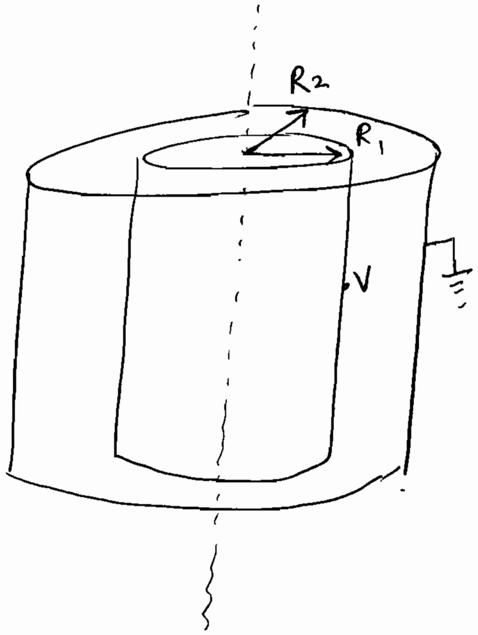
$$E = \frac{Q_e}{4\pi\varepsilon_0 r^2} \Rightarrow V(R_2) - V(r) = \frac{-Q_e}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{R_2}\right)$$

$$\Rightarrow +V = \frac{Q_e}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{R_2}\right)$$

@ $r = R_1$, $V = V_1 \Rightarrow Q_e = \frac{V_1 4\pi\varepsilon_0}{\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$

$$\Rightarrow \boxed{V = \frac{V_1}{\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}\left(\frac{1}{r} - \frac{1}{R_2}\right)}$$

(6)



Conducting cylinders $\Rightarrow$ Charge in between = 0

These concentric spheres, cylinders etc. form Capacitance. The Potentials in between are used to stored charge. Similarly 2 parallel plates are used to store charge.

Cylindrical

$$\boxed{\nabla^2 \lambda(r, \theta, z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \lambda}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \lambda}{\partial \theta^2} + \frac{\partial^2 \lambda}{\partial z^2}}$$

Looking at geometry of problem, $V = f(r)$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0$$

$$\Rightarrow r\frac{dV}{dr} = A \Rightarrow \frac{dV}{dr} = \frac{A}{r} \Rightarrow \boxed{V = A\ln r + B}$$

We can find out A and B via Boundary Conditions.

→ Can be verified from gauss law assuming Q_{enc}

@ r_2 : $V = 0$

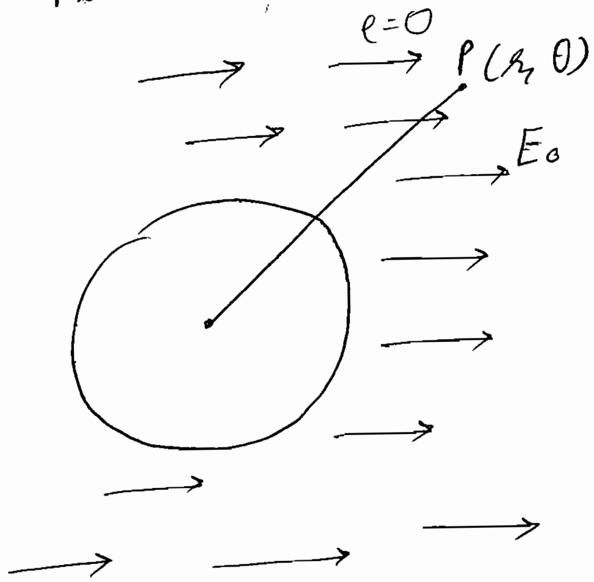
$$V = A \ln R_2 + B = 0 \Rightarrow B = -A \ln R_2$$

$$\Rightarrow \boxed{V = A \ln\left(\frac{r}{R_2}\right)}$$

@ r_1, $V = V_1$ $\Rightarrow$ $\boxed{V = \frac{V_1}{\ln\left(\frac{R_2}{R_1}\right)} \ln\left(\frac{R_2}{r}\right)}$

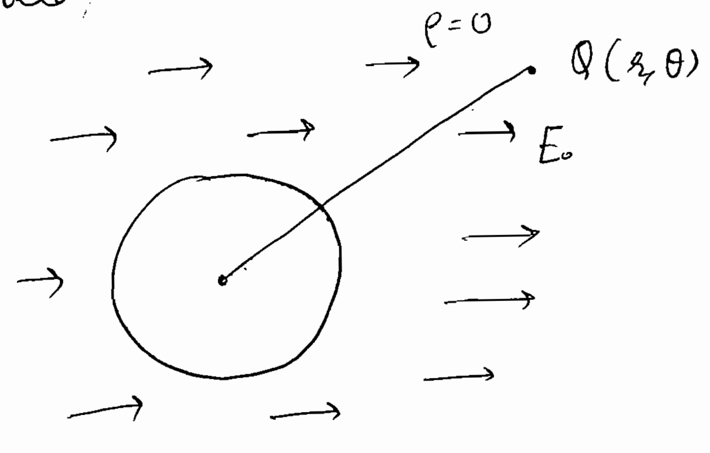
⊛ Till now, we have dealt with simple problems of single dimension. Boundary Value Problems are difficult as they deal with multiple dimension, hence separation of variables are used!!

(v) Boundary Value Problem of Conducting Sphere in Uniform Field $\vec{E_0}$



Find V_P and E_P

(vi) Boundary Value Problem of Dielectric Sphere in Uniform Field



Find V_Q and E_Q.

For conducting sphere placed in E_0.

$\rho_{outside} = 0$

$\rho_{inside} = 0$

$V_{inside} = V_{surface}$

$\nabla^2 V = 0$

→ Before solving specifically for a conductor or dielectric, let us find the general solution of laplace equation in spherical coordinates.

Specific solutions will be nothing but Boundary Values.

(★) $V = f(r, \theta)$

Conducting Sphere

$$\nabla^2 V = 0 \qquad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)$$

$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

$\boxed{\text{Separation of variables}}$ are used to solve such types of 3-d problems.

$$V(r, \theta, \phi) = R_{(r)}\, \Theta_{(\theta)}\, \Phi(\phi) \qquad \text{(say)}$$

$$\frac{\partial^2 V}{\partial r^2} = \Theta \Phi \left(\frac{d^2 R}{dr^2}\right)$$

$$\frac{\partial^2 V}{\partial \theta^2} = R \Phi \left(\frac{d^2 \Theta}{d\theta^2}\right)$$

$$\frac{\partial^2 V}{\partial \phi^2} = \left(\frac{d^2 \Phi}{d\phi^2}\right) R \Theta$$

done on Book: P-158

We are looking for solutions, that are products of functions, each of which depends on only one of the coordinates !!

Multiplying by $r^2 \sin^2\theta$,

$$\sin^2\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \sin\theta \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{\partial^2 V}{\partial \phi^2} = 0$$

Separating the variables,

$$\Rightarrow \Theta \Phi \sin^2\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \cdots$$

Divide by $R \Theta \Phi$, we get

$$\Rightarrow \frac{1}{R} \sin^2\theta \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{\Theta} \sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right)$$

$$= \underbrace{-\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2}}_{} = m^2 \text{ (say)}$$

it has to be a const. as LHS is not a function of ϕ

$$\Rightarrow \frac{d^2\Phi}{d\phi^2} = -\Phi m^2$$

$$\Rightarrow \boxed{\Phi = e^{\pm i m \phi} = C_1 \sin m\phi + C_2 \cos m\phi}$$

dividing by $\sin^2\theta$, ★ Assuming azimuthal symmetry: that V is not a function of ϕ s.t. $m = 0$.

$$\frac{1}{R} \frac{\cancel{\sin^2\theta}}{\cancel{\sin^2\theta}} \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = -\frac{1}{\Theta} \frac{\sin\theta}{\sin^2\theta} \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right)$$

Since both are equated $\Rightarrow$ they are pure numbers

Let us take const. = $n(n+1)$ where n is an integer.

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = n(n+1)$$

→ Here we cannot assume integer $n \in \mathbb{R}^+$

$$\Rightarrow \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = +R\,n(n+1)$$

$$\Rightarrow r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} = nR(n+1)$$

→ From Power Series solution, we will get to know that n is an integer

Let $\boxed{R = r^\alpha}$ (or $e^{\alpha r}$)

$$\frac{dR}{dr} = \alpha r^{\alpha-1}$$

$$\frac{d^2R}{dr^2} = \alpha(\alpha-1)\,r^{\alpha-2}$$

$$\Rightarrow r^2\alpha(\alpha-1)r^{\alpha-2} + 2r\alpha r^{\alpha-1} = n r^\alpha(n+1)$$

$$\Rightarrow r^\alpha\left[\alpha^2+\alpha - n(n+1)\right] = 0$$

that's why we took $n(n+1)$

$$\Rightarrow r^\alpha\left[\alpha(\alpha+1) - n(n+1)\right] = 0$$

$$\Rightarrow \boxed{\alpha = n} \text{ or } \boxed{\alpha = -(n+1)}$$

$$\alpha = \frac{-1\pm\sqrt{1+4n(n+1)}}{2}$$

$$= \frac{-1\pm\sqrt{4n^2+4n+1}}{2}$$

$$= \frac{-1\pm(2n+1)}{2}$$

$$= n, -(n+1)$$

$\Rightarrow$

$$\Rightarrow \boxed{R = A_n r^n + B_n r^{-(n+1)}}$$

(*) Note that assumption of azimuthal symmetry is almost always valid wherever $\vec{E}$ is taken along z-axis.

$$\frac{1}{\Theta}\,\frac{1}{\sin\theta}\,\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) = -n(n+1)$$

do not ignore this minus sign.

$$\Rightarrow \boxed{\frac{1}{\sin\theta}\,\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) + n\Theta(n+1) = 0}$$

Legendre's Differential Equation

Take $\cos\theta = x$

$$-\sin\theta\, d\theta = dx$$

• note that कि अगर m लेंगे तो associate legendre आएगी और अगर m नहीं लेंगे तो legendre आएगी !!

$$\frac{d}{d\theta} = \frac{d}{dx}\cdot\frac{dx}{d\theta} = -\sin\theta\,\frac{d}{dx}$$

$$\frac{1}{\sin\theta}\,\frac{d}{d\theta} = -\frac{d}{dx}$$

Both for both, eigen value निकालने के लिए m नहीं लेना है!!

$$\frac{d}{dx}\left[-\sin^2\theta\,\frac{d\Theta}{dx}\right] + n(n+1)\,\Theta = 0$$

$$\Rightarrow \frac{d}{dx}\left[(1-x^2)\,\frac{d\Theta}{dx}\right] + n(n+1)\,\Theta = 0$$

$$\Rightarrow \boxed{(1-x^2)\,\frac{d^2\Theta}{dx^2} - 2x\,\frac{d\Theta}{dx} + n(n+1)\,\Theta = 0}$$

Solved by: $\Theta_{(x)} = \sum_{k=0}^{\infty} a_k x^k$ Power Series Solution

Solution: $\Theta = \Theta_n(x)$: legendre's polynomials

or

$$\boxed{P_n(x) = \frac{1}{2^n \lfloor n}\,\frac{d^n}{dx^n}(x^2-1)^n}$$

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{3x^2-1}{2}$$

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{(3x^2-1)}{2}$$
$$\vdots$$

$$\boxed{\begin{aligned} V_{r,\theta,\phi} &= R_{(r)}\,\Theta_{(\theta)}\,\Phi(\phi) \\ &= \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta)\; e^{\pm im\phi} \end{aligned}}$$

- Now we need to determine A_n and B_n using Boundary Conditions !!

(★) $\vec{E}$ is a vector quantity having 3 components when V is a scalar quantity. Yet $\vec{E} = -\vec{\nabla} V$

This is extraordinary because 1 scalar function contains information about all the 3 components. Answer is that components of E are not independent but connected by three equations: $\vec{\nabla} \times \vec{E} = 0$

Therefore, in reality what Poisson equation does is to reduce a vector problem into 1 scalar problem in which there is no need to fuss with the components.

(★) We usually use ∞ as reference point. s.t. $V_\infty = 0$ but this might not always work.

eg $\left[V(z) \text{ for a plane} = -\int_{\infty}^{z} \frac{1}{2\varepsilon_0}\sigma \cdot dz = \frac{\sigma}{2\varepsilon_0}(z-\infty) \right]$

Remedy is simply to choose any other point as reference point eg. we might choose the origin.

(★) Only thing worth knowing for conductors is that $\vec{E}_{inside} = 0$ and $V_{inside} = V_{surface} = \text{Const.}$

Vector Calculus of Curvilinear Coordinates

H.W.

(Appendix A : Griffith)

- $dl = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$: Cartesian
- $dl = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$: Spherical
- $dl = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$: Cylindrical

- Writing in terms of Coefficient

	C_x	C_y	C_z
Cartesian	1	1	1
spherical	1	r	$r\sin\theta$
cylindrical	1	s	1

such that for a general coordinate system,

$$\boxed{\vec{dl} = C_x\,dx\,\hat{x} + C_y\,dy\,\hat{y} + C_z\,dz\,\hat{z}}$$

Gradient

We know $dT = \vec{\nabla}T \cdot \vec{dl}$

$$= (\vec{\nabla}T)_x\,C_x\,dx + (\vec{\nabla}T)_y\,C_y\,dy + (\vec{\nabla}T)_z\,C_z\,dx$$

$$\Rightarrow \boxed{\vec{\nabla}T = \frac{1}{C_x}\left(\frac{\partial T}{\partial x}\right)\hat{x} + \frac{1}{C_y}\left(\frac{\partial T}{\partial y}\right)\hat{y} + \frac{1}{C_z}\left(\frac{\partial T}{\partial z}\right)\hat{z}}$$

V_{dipole} से $\vec{E}_{dipole}$ निकालना।

Divergence

$$\boxed{\vec{\nabla}\cdot\vec{A} = \frac{1}{C_x C_y C_z}\left[\frac{\partial}{\partial x}(C_y C_z A_x) + \frac{\partial}{\partial y}(C_z C_x A_y) + \frac{\partial}{\partial z}(C_x C_y A_z)\right]}$$

Curl

$$\vec{\nabla}\times\vec{A} = \frac{1}{C_y C_z}\left[\frac{\partial}{\partial y}(C_z A_z) - \frac{\partial}{\partial z}(C_y A_y)\right]\hat{x}$$

$$+ \frac{1}{C_z C_x}\left[\frac{\partial}{\partial z}(C_x A_x) - \frac{\partial}{\partial x}(C_z A_z)\right]\hat{y}$$

$$+ \frac{1}{C_x C_y}\left[\frac{\partial}{\partial x}(C_y A_y) - \frac{\partial}{\partial y}(C_x A_x)\right]\hat{z}$$

$$\boxed{\Leftrightarrow \frac{1}{C_x C_y C_z}\begin{vmatrix} C_x\hat{x} & C_y\hat{y} & C_z\hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ C_x A_x & C_y A_y & C_z A_z \end{vmatrix}}$$

★ 1^{st} और 3^{rd} row में coefficients multiplied हैं और $\frac{1}{C_x C_y C_z}$ to of course बाहर है

laplacian $\Leftrightarrow \vec{\nabla}\cdot * \vec{\nabla}$

Laplacian

$$\nabla^2 T = \nabla\cdot\vec{\nabla}T = \frac{1}{C_x C_y C_z}\left[\frac{\partial}{\partial x}\left(\frac{C_y C_z}{C_x}\left(\frac{\partial T}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(\frac{C_z C_x}{C_y}\left(\frac{\partial T}{\partial y}\right)\right) + \frac{\partial}{\partial z}\left(\frac{C_x C_y}{C_z}\left(\frac{\partial T}{\partial z}\right)\right)\right]$$

$\longrightarrow$ H-atom problem

E&M (4)

17/02/2012

- For conducting sphere placed in Electric Field,

$$\boxed{V_{(r,\theta)} = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos\theta)}$$

where $P_n(x) = \dfrac{1}{2^n \lfloor n} \dfrac{d^n}{dx^n} (x^2-1)^n$

(*) THESE ARE STANDARD RESULTS OF SOLUTION OF A LAPLACE EQUATION IN SPHERICAL COORDINATES.

Also $\displaystyle\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0$ for $m \neq n$

- Legendre Polynomials are orthogonals i.e. Cross* product = 0
  *inner

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \frac{2}{2n+1} \quad \text{for } m = n$$

Hence Orthonormal Polynomials

Q/ 24 & 26 :- same derivation for separation of variables

Boundary Value Problem of Conducting Sphere in Uniform Field (Assume field in z-axis)

(i) at $r = R$, V is equipotential i.e. $V = \text{const.} = 0$ [by suitable choice of scale]

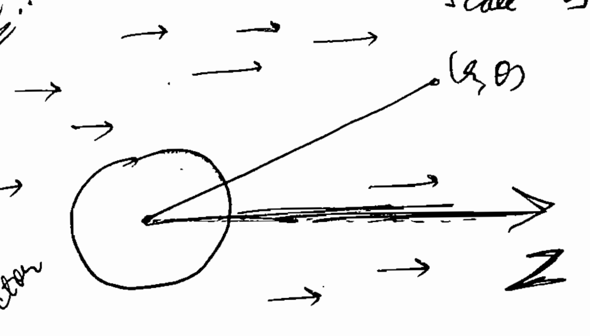
(ii) at $r \gg R$, $V = -\displaystyle\int_0^r E_0 \, dx$ = ~~...~~ $= E_0 \cdot r\cos\theta$ ✓ correct!!

ONLY 2 CONDITIONS

Potential @ O = O as complete volume inside sphere is equipotential

[Potential drops along direction of $\vec{E}$]

taking effect of E_0 and not conductor



$$V = \left(A_0 + \frac{B_0}{r}\right) + \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta + \cdots$$

From (ii)

looking @ Boundary Value, $V = -E_0 r\cos\theta$

hence retaining only 2 terms, because 3 and above will have $\cos^2\theta$ dependence !!

$$\Rightarrow V = \left(A_0 + \frac{B_0}{r}\right) + \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta$$

Note that here we are not looking for any other possibilities. We just want 1 function that satisfied B.C. Uniqueness Theorem will ensure that there are no other solutions

@ $r \gg R$ $\quad V = A_0 + A_1 r\cos = -E_0 r\cos\theta$

$$\begin{cases} A_1 = -E_0 \\ A_0 = 0 \end{cases}$$

From (i)

$$V_R = 0 = \frac{B_0}{R} + \left(-E_0 R + \frac{B_1}{R^2}\right)\cos\theta$$

Comparing coefficients, (Note that we can vary θ and it will still be 0)

$B_0 = 0$

$$\frac{B_1}{R^2} - E_0 R = 0 \Rightarrow B_1 = +E_0 R^3$$

$$E_0 R^3 \Leftrightarrow \frac{p}{4\pi\epsilon_0}$$

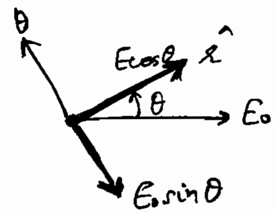
$$\Rightarrow \boxed{V(r,\theta) = -E_0 r\cos\theta + \frac{E_0 R^3}{r^2}\cos\theta}$$

Original field : $E_0\cos\theta\,\hat{r} - E_0\sin\theta\,\hat{\theta}$

$$\vec{E} = E_r\hat{r} + E_\theta\hat{\theta}$$

$$E_r = -\left(\frac{\partial V}{\partial r}\right) = E_0\cos\theta + \frac{2E_0 R^3\cos\theta}{r^3}$$

$$E_\theta = -\frac{1}{r}\left(\frac{\partial V}{\partial \theta}\right) = -E_0\sin\theta + \frac{E_0 R^3}{r^3}\sin\theta$$



$$\boxed{\text{Modified } \vec{E} = \frac{E_0 R^3}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]}$$

We can also take modified Potential and find out $\vec{E}_{modified}$

Hence it behaves like a dipole.

Note that modified field falls off as $\left(\frac{1}{r^3}\right)$, hence at $r \gg R$, only effect of E_0

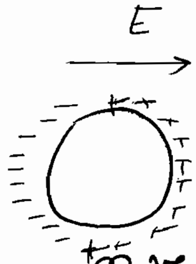
Surface Charge Density σ

We know @ surface of conductor,

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}$$

For sphere $\vec{E} = \frac{\sigma}{\epsilon_0}\hat{r} = \vec{E_r}$

$$\frac{\sigma}{\epsilon_0} = E\cos\theta + 2E_0\cos\theta\frac{R^3}{R^3} = 3E_0\cos\theta$$



$$\Rightarrow \boxed{\sigma = 3\epsilon_0 E_0\cos\theta}$$

{Upper surface: ⊕ve charge
lower surface: ⊖ve charge}

$$Q = \iint \sigma\, dA = \iint \sigma\, r^2\sin\theta\, d\phi\, d\theta$$

Q should come out to be zero because it behaves like a dipole. Also it is induced charge, net charge conserved. hence $\boxed{Q = 0.}$

NOTE THAT CONDUCTOR के अंदर का ANALYSIS नहीं करना !!

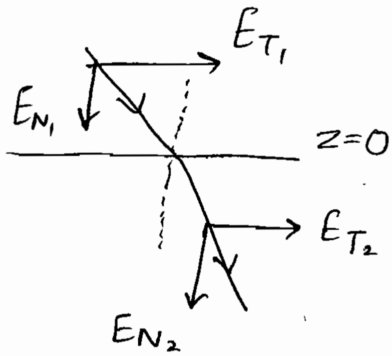
- At $r = R$, Note that $E_\theta = 0$ ✓
  $E_r = 3E_0\cos\theta$ ✓
- For Conductor, $V_{in} = V_{surface}$ ✓

Boundary Value problem with dielectric sphere placed in $\vec{E}$

Dielectric Sphere

Boundary Conditions

At interface of 2 dielectric media, tangential vector of $\vec{E}$ remains conserved.



① $\boxed{E_{t_1} = E_{t_2}}$ [NO Force acting hence Tangential Component remains conserved]

&

② $\epsilon_1 E_{n_1} = \epsilon_2 E_{n_2}$

i.e. $k_1 \epsilon_0 E_{n_1} = k_2 \epsilon_0 E_{n_2}$

$\Rightarrow k_1 E_{n_1} = k_2 E_{n_2}$

$$\boxed{k_1 \left(\frac{\partial V}{\partial r}\right)_1 = k_2 \left(\frac{\partial V}{\partial r}\right)_2}$$

$\boxed{D_{n_1} = D_{n_2}}$

→ Due to Polarization in dielectrics we do not know q_{enc} due to bound charges ∴ cannot apply normal gauss law

→ Applying Gauss law for dielectrics

$\vec{\nabla} \cdot \vec{D} = \rho_f \Rightarrow \oint \vec{D} \cdot d\vec{s} = q_f$

$\Rightarrow D_1 \cdot A - D_2 A = 0 \Rightarrow \boxed{D_1 = D_2}$

let $V_{in} = V_{r<R}$

$V_0 = V_{r>R}$

• Do not worry about bound charges & Laplace Equation

$\vec{E} = -\vec{\nabla} V \Rightarrow \vec{D} = -\varepsilon \vec{\nabla} V$ (linear)

$\nabla \cdot D = \rho_f \Rightarrow -\varepsilon \nabla^2 V = \rho_f = 0$

$\Rightarrow \boxed{\nabla^2 V = 0}$

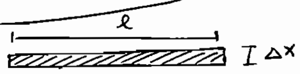
③ $\nabla^2 V_{in} = 0$

$$\Rightarrow V_{in} = \sum \left(A_n' r^n + \frac{B_n'}{r^{n+1}}\right) P_n(\cos\theta)$$

④ $\nabla^2 V_{out} = 0$

$$V_0 = \sum \left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta)$$

Proof of $E_{T_1} = E_{T_2}$



$\Delta x \to 0$

$\vec{\nabla} \times \vec{E} = 0$

$\Rightarrow \int E \cdot dl = 0$

$\Rightarrow E_{T_1} l + E_{n_1} \Delta x - E_{T_2} l - E_{n_2} \Delta x = 0$

$\Rightarrow E_{T_1} l - E_{T_2} l = 0$

$\Rightarrow \boxed{E_{T_1} = E_{T_2}}$

⑤ @ $r = R$ $\quad V_i = V_0$

⑥ @ $r = R$ $\quad k_1 E_{n_1} = k_2 E_{n_2}$

i.e. $$k_1 \left(\frac{\partial V_i}{\partial r}\right)\Bigg|_{r=R} = k_2 \left(\frac{\partial V_o}{\partial r}\right)\Bigg|_{r=R}$$

(7) $r >> R$ $\quad V_o = -E_o r \cos\theta$

ONLY 4 CONDITIONS (5, 6, 7, 8)

(8) $r = 0$ $\quad V_{in} =$ finite

$\Rightarrow B_n'$ are 0

$$\Rightarrow \boxed{V_{in} = \sum \left(A_n' r^n + \frac{B_n'}{r^{n+1}}\right) P_n(\cos\theta)}$$

$$\Rightarrow V_{in} = \sum A_n' r^n P_n(\cos\theta)$$ Using (8)

n will go upto 0 and 1

$$\Rightarrow V_{in} = A_0' P_0(\cos\theta) + A_1' r P_1(\cos\theta)$$

$$\Rightarrow \boxed{V_{in} = A_0' + A_1' r\cos\theta}$$ ✓

Note that basic expressions are same. Only difference is due to different boundary conditions & hence different constants.

★★ $\boxed{\text{Choosing } V_{in0} = A_0' = 0}$

$$\Rightarrow \boxed{V_{in} = A_1' r\cos\theta}$$

Now $$\boxed{V_0 = \sum \left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta)}$$

· @ $r >> R$ $$-E_o r\cos\theta = \sum A_n r^n P_n(\cos\theta) + \frac{B_0}{r} + \frac{B_1\cos\theta}{r^2}$$

$$= \left(A_0 + A_1 r\cos\theta + \frac{B_0}{r} + \frac{B_1}{r^2}\cos\theta\right)$$ [only dependence till $\cos\theta$]

Using (7) $\Rightarrow A_0 = 0$ (comparing coefficients)

$\Rightarrow A_1 = -E_0$

$$\Rightarrow \boxed{V_o = -E_o r\cos\theta + \frac{B_o}{r} + \frac{B_1}{r^2}\cos\theta}$$

Using (5) at $r=R$: $A_o' + A_1' r\cos\theta = -E_o r\cos\theta + \frac{B_o}{r} + \frac{B_1}{r^2}\cos\theta$

Comparing Coefficients

$\Rightarrow B_o = 0$ (since $A_o' = 0$)

$$\boxed{A_1' = -E_o + \frac{B_1}{R^3}}$$

@ $r=R$ $V_{in} = V_{out}$ & $E_{t_1} = E_{t_2}$ give same equation!!

Using (6) For radial coordinates, for a sphere $\left(\frac{\partial V}{\partial n}\right) = \left(\frac{\partial V}{\partial r}\right)$

@ $r=R$ $k_1\left(\frac{\partial V_i}{\partial r}\right) = k_2\left(\frac{\partial V_o}{\partial r}\right)$

always remember this step....

$\Rightarrow k_1 A_1'\cos\theta = k_2\left(-E_o\cos\theta - 2\frac{B_1}{r^3}\cos\theta\right)$

$$\Rightarrow \boxed{k_1 A_1' = -k_2 E_o - \frac{2k_2 B_1}{R^3}}$$

Solving the 2 eqns,

we get V, E, induced charge

$$-k_1 E_o + \frac{k_1 B_1}{R^3} = -k_2 E_o - \frac{2k_2}{R^3}B_1$$

$$\Rightarrow \boxed{B_1 = \frac{(k_1-k_2)E_o R^3}{(k_1+2k_2)}}$$

$$A_1' = -E_o + E_o\frac{(k_1-k_2)}{(k_1+2k_2)}$$

$$\boxed{A_1' = \frac{-3k_2 E_o}{(k_1+2k_2)}}$$

$$\Rightarrow \boxed{\begin{aligned} V_i &= \frac{-3k_2 E_0 r \cos\theta}{(k_1+2k_2)} \\ V_o &= -E_0 r\cos\theta + \frac{(k_1-k_2)}{(k_1+2k_2)} \frac{E_0 R^3}{r^2}\cos\theta \end{aligned}}$$

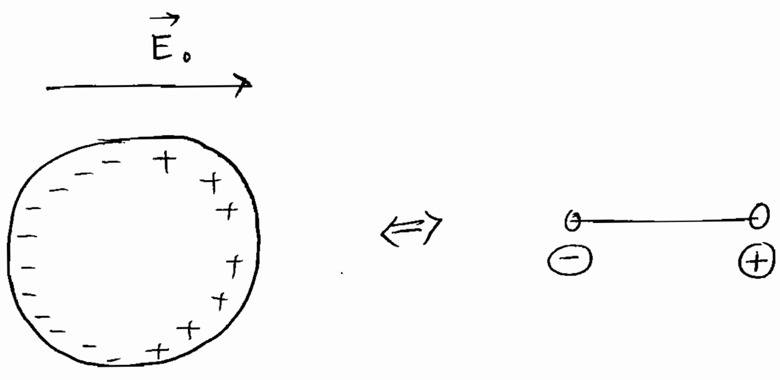
Dielectric sphere is behaving like a dipole.

Modified Potential $V_{m\,(\text{outside})} = \frac{(k_1-k_2)E_0 R^3}{(k_1+2k_2)} \frac{\cos\theta}{r^2}$

Taking anology with dipole potential

Hence $\frac{p}{4\pi\epsilon_0} = \frac{(k_1-k_2)}{(k_1+2k_2)} E_0 R^3$

$$\Rightarrow \boxed{p_{\text{equivalent}} = \frac{4\pi\epsilon_0 (k_1-k_2)}{(k_1+2k_2)} E_0 R^3}$$



Using volume as $\frac{4}{3}\pi R^3$ and using Polarization as dipole moment per unit volume,

$$\boxed{\vec{P} = \frac{p}{\frac{4}{3}\pi R^3} = 3\epsilon_0 E_0 \left(\frac{k_1-k_2}{k_1+2k_2}\right)}$$

- Due to displacement of charge inside dielectric, caused due to applied external field, causes Polarization.

$$\cancel{A_r = 0 \quad \vec{E} = -\vec{\nabla} V}$$

$$\cancel{E_r = \frac{\sigma}{\epsilon_0} = \frac{-2k_2 \ldots}{(k_1 + 2k_2)} E_0 \cos\theta}$$

For cylindrical Coordinates, solving Laplace Equation :

$$V = V(r, \phi, z)$$

$$\nabla^2 V = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \left(\frac{\partial^2 V}{\partial z^2}\right) = 0$$

$$V(r, \phi, z) = R(r)\, \Phi(\phi)\, Z(z)$$

$\nabla^2 V = 0$ and dividing by $R \Phi Z$

$$\Rightarrow \frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r \frac{dR}{dr}\right) + \frac{1}{\Phi r^2}\left(\frac{d^2 \Phi}{d\phi^2}\right) + \frac{d^2 Z}{dz^2} = 0.$$

$$\frac{d^2 Z}{dz^2} + m^2 Z = 0$$

$$\Rightarrow \boxed{Z = e^{\pm imz}}$$

Making $m = 0$, for 2 coordinates i.e. assuming cylindrical symmetry and no dependence on z

$$\frac{1}{R} r \frac{d}{dr}\left[r \frac{dR}{dr}\right] = -\frac{1}{\Phi}\frac{d^2 \Phi}{d\phi^2} = n^2 \text{ (say)}$$

$$-\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = n^2 \Rightarrow \qquad \Phi = e^{\pm in\phi}$$

$$\boxed{\Phi(\varphi) = \sum A_n \cos n\phi + B_n \sin n\phi}$$

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) = n^2 R$$

take $R = r^{\alpha}$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} = n^2 R$$

$$\Rightarrow r^2\alpha(\alpha-1)r^{\alpha-2} + r\alpha r^{\alpha-1} = n^2 r^{\alpha}$$

$$\Rightarrow \alpha(\alpha-1) + \alpha = n^2$$

$$\Rightarrow \alpha = \pm n$$

$$\boxed{R(r) = \sum_{n=0}^{\infty} C_n r^n + D_n r^{-n}}$$

Its not taking into account for $n=0$

Handling $n=0$ Case separately ....

$$\Rightarrow r\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

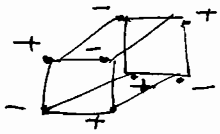
[Put this form as 0 and not the opened form. फिजूल में [दिमाग खराब होगा]]

$$\Rightarrow r\frac{dR}{dr} = A \qquad \Rightarrow R = A\ln r + B$$

Hence $R(r) = A\ln r + B + \sum_{n=1}^{\infty}(C_n r^n + D_n r^{-n})$

★ At large distances, potentials fall off as

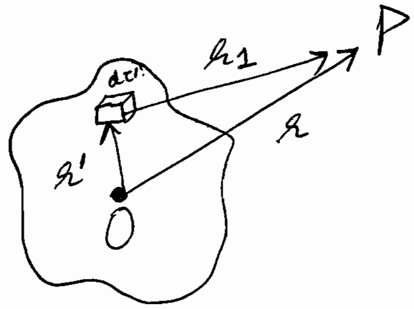
Monopole $\left(V \sim \frac{1}{r}\right)$ Dipole $\left(V \sim \frac{1}{r^2}\right)$ Quadrupole $\left(V \sim \frac{1}{r^3}\right)$



Octapole $\left(V \sim \frac{1}{r^4}\right)$

★ From Multipole Expansion,

$$V(r) = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\theta')\, \rho(\vec{r}')\, d\tau'$$



For monopole, placing charge at Origin O,

$$V_{monopole}(r) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r}$$

For a point charge at the origin, $V_{monopole}(r)$ represents the exact potential everywhere, not merely a first approximation at a large r.
(in this case, potential due to higher terms vanish)
If point charge is not placed at origin, then $V(r)$ consists of other terms also.
But note that for a dipole, the higher terms do not vanish but become progressively smaller. So the formula

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{r^2}\, \hat{r} \cdot \int \vec{r}'\, \rho(r')\, d\tau'$$

$$V_{dipole}(r) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} \int r'\, \rho(r')\, d\tau' \cos\theta' = \frac{1}{4\pi\varepsilon_0} \frac{p\cos\theta}{r^2}$$

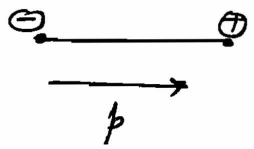
(Refer PTO PTO)

represent only on the 1st non zero term & is not exact, therefore.

Of course, as we go further and further away from the dipole, the $V_{dipole}(r)$ becomes better and better approximation. By the same token, at a fixed r, the dipole approximation improves as you shrink the separation d. To construct a dipole whose potential is exactly given by $V_{dipole}(r)$, we will have have let $d \to 0$. Correspondingly to conserve p, we will need to let $q \to \infty$ i.e. A physical dipole becomes a pure dipole in the artificial limit $q \to \infty$ and $d \to 0$.

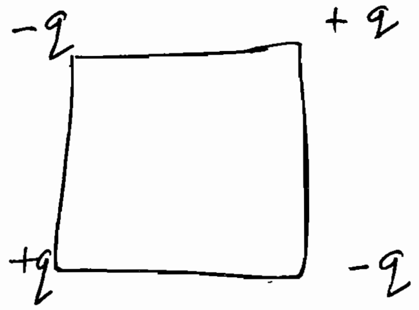
BUT FOR ALL PRACTICAL PURPOSE, WE USE $V_{dipole}(r)$ FOR PHYSICAL DIPOLES WITH SMALL BUT FINITE separation d.

(*) Dipoles add like vectors.



$$\vec{p}_{total} = \sum_i \vec{p}_i$$

eg

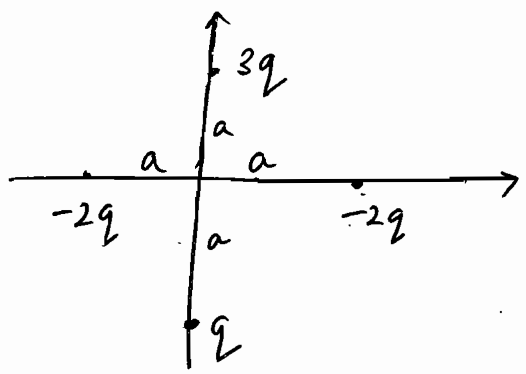


$$\vec{p}_{total} = \vec{p}_1 + \vec{p}_2 = 0$$

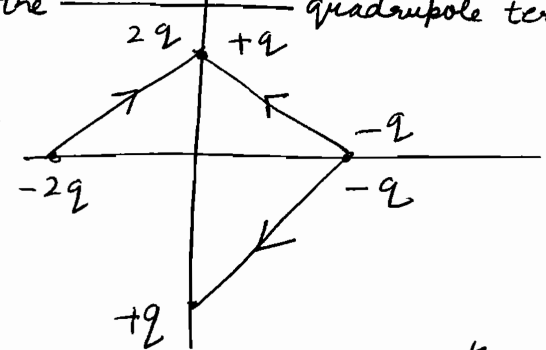
$\longrightarrow \quad \longleftarrow$

→ This is a quadrupole and potential is dominated by quadrupole term in multipole Expansion.

So, if all dipoles add up to zero vectorially, we can directly write the quadrupole term.



$\Longleftrightarrow$

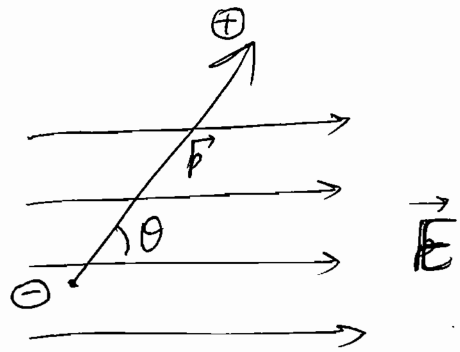


We can now add them vectorially.

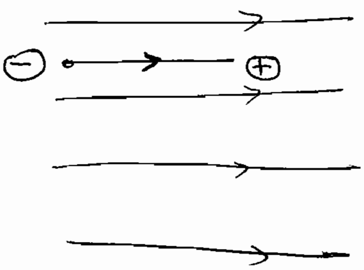
(*) Multipole Expansion depends drastically on choice of origin.

Potential Energy for a dipole

We can see $\tau = \vec{p} \times \vec{E}$



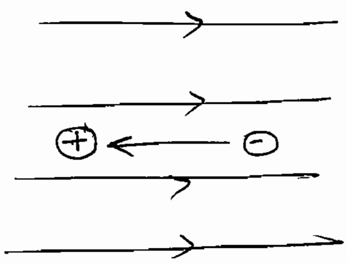
Now consider



We can see that it is stable equilibrium

$\Rightarrow$ U_{min} at $\theta = 0$

Now consider



We can see that it is in Unstable equilibrium

$\Rightarrow$ U_{max} at $\theta = 180°$

Also $U_{max} - U_{min} = \int_0^{\pi} \tau_{ext} \, d\theta = +pE\int_0^{\pi} \sin\theta \, d\theta$

$= 2pE$

$$\boxed{\begin{aligned} \tau &= -pE\sin\theta \\ \tau_{ext} &= pE\sin\theta \\ \Rightarrow dW &= pE\sin\theta\, d\theta \\ \Rightarrow W &= -pE\cos\theta\Big|_0^{\pi} \\ &= pE.2 \\ &= 2pE \end{aligned}}$$

Choose $U_{max} = pE$

$U_{min} = -pE$

Now $U_f - (-pE) = \int_0^{\theta} pE\sin\theta \, d\theta$

$= pE.(1-\cos\theta)$

$\Rightarrow U_f = -pE\cos\theta$

$\Rightarrow$ $\boxed{U(\theta) = -\vec{p}\cdot\vec{E}}$

⊛ Torque on a dipole in non uniform field :

$$N = \vec{p} \times \vec{E} + \vec{r} \times \vec{F}$$

where $\vec{F} = (\vec{p} \cdot \vec{\nabla}) \vec{E}$

⊛ Also from Multipole Expansion, the dipole term is:

$$V_{dipole} = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \int r' \cos\theta' \rho(\vec{r}') d\tau'$$

Now $r' \cos\theta' = \vec{r}' \cdot \hat{r}$

$$\Rightarrow V_{dipole} = \frac{1}{4\pi\epsilon_0} \cdot \frac{\hat{r}}{r^2} \cdot \int \vec{r}' \rho(\vec{r}') d\tau'$$

We define dipole moments as $\underline{\underline{\vec{p} = \int \vec{r}' \rho(\vec{r}') d\tau'}}$

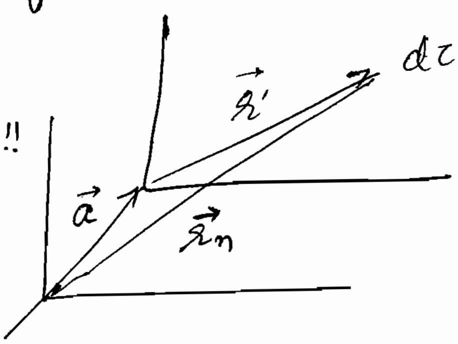
$$\Rightarrow V_{dipole} = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2}$$

Lets note :

(1) A point charge placed at Origin Constitutes a "pure" monopole. If its not at Origin, then other terms of multipole expansion will also appear.

(2) The dipole moment $\vec{p}$ does not change when we shift the Origin only if net charge is zero. Otherwise it depends upon choice of Origin

$$\vec{p}_{new} = \int \vec{r}_n \underbrace{\rho(\vec{r}') d\tau'}_{dq \text{ तो same ही रहेगा!!}}$$



$$= \int (\vec{r}' + \vec{a}) \rho(r') d\tau'$$

$$= \int \vec{r}' \rho(r') d\tau' + \vec{a} \int \rho(r') d\tau'$$

$$= \underline{\vec{p} + \vec{a} Q}$$

E & M (5)

18/1/2012

For cylindrical coordinates, solving Laplace gives :-

$$\boxed{V(r,\phi,z) = \left[(A\ln r + B) + \sum_{n=1}^{\infty}(C_n r^n + D_n r^{-n})\right]\cdot\left[\sum A_n \cos n\phi + B_n \sin n\phi\right]\left[e^{\pm imz}\right]}$$

Dipole

Single charge ~~is~~ called Monopole (1) $l=0$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r}\right)$$

If there are 2 charges forming a dipole (2) $l=1$

$$\boxed{V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}} \checkmark$$

Quadrupole (4) $l=2$

Octapole (8) $l=3$

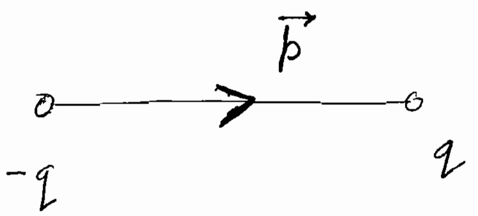
Hexapole (16) $l=4$

In general 2^l poles in Multipole System.

- If no $\vec{E}_{external}$, all multipoles are arranged randomly. When $\vec{E}_{ext}$ is applied, charges are displaced and multipoles are formed.

Dipole

- Equal & opposite charges separated by atomic dimensions.

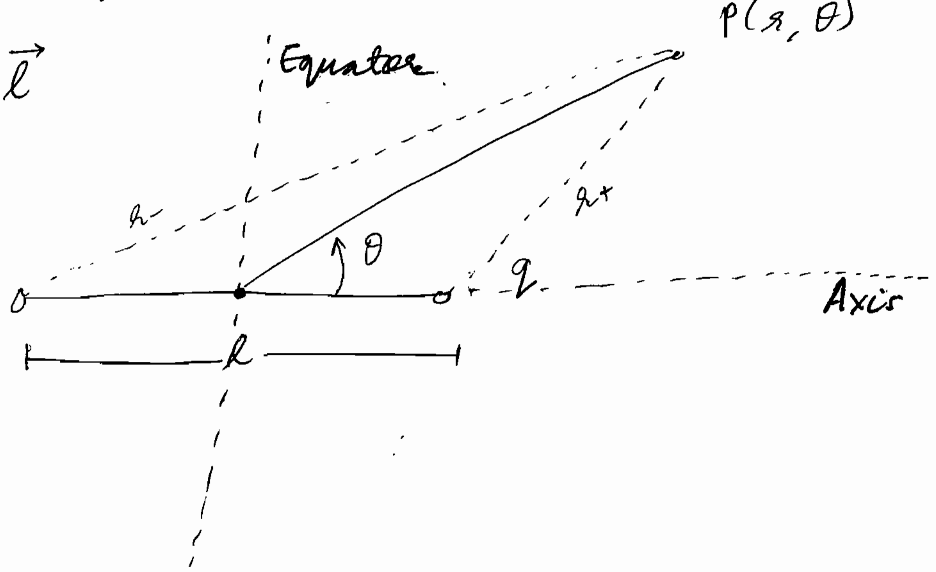


- $\vec{p}$ from $-q$ to $+q$.

$\hat{\theta}$ $\hat{r}$

$\vec{p} = q\vec{l}$

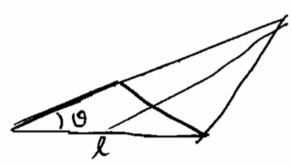
- Let us consider



Note that $l << r$. $\Rightarrow r_+ \simeq r_-$

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r_+} - \frac{q}{r_-}\right) = \frac{q}{4\pi\epsilon_0}\left[\frac{r_- - r_+}{r_+ r_-}\right]$$

Approximating $\angle P q_- q_+ = \theta$



✓ $(r_-) - (r_+) \simeq l\cos\theta$

✓ $r_- r_+ \simeq r^2$

$$\Rightarrow V(r,\theta) = \frac{q}{4\pi\epsilon_0}\frac{l\cos\theta}{r^2} = \frac{1}{4\pi\epsilon_0}\left(\frac{p\cos\theta}{r^2}\right)$$ ✓

✓ (Note that in diffraction, we used to measure $(90^\circ - \theta)$. $\therefore \sin\theta$ there)

$$\vec{E} = -\vec{\nabla} V$$

$$= -\left[\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}\right]$$

$$= +\frac{2p\cos\theta}{4\pi\epsilon_0 r^3}\hat{r} + \frac{p\sin\theta}{4\pi\epsilon_0 r^3}\hat{\theta}$$

$$\boxed{\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]}$$

Note that at equitorial point, $V = 0$

Note that $V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$

$$\vec{E} = \frac{-1}{4\pi\epsilon_0}\vec{\nabla}\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right)$$

Take 1 scalar as $\vec{p}\cdot\vec{r}$, other scalar as $\left(\frac{1}{r^3}\right)$

We know $\boxed{\vec{\nabla}(fg) = (\vec{\nabla}f)g + f(\vec{\nabla}g)}$

$$\Rightarrow \vec{E} = -\frac{1}{4\pi\epsilon_0}\left[\frac{1}{r^3}\vec{\nabla}(\vec{p}\cdot\vec{r}) + \vec{p}\cdot\vec{r}\,\vec{\nabla}\left(\frac{1}{r^3}\right)\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\left[\frac{1}{r^3}\cdot[\vec{p}] + \vec{p}\cdot\vec{r}\left[\frac{-3\vec{r}}{r^5}\right]\right]$$

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\vec{p}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{p}}{r^3}\right]}$$

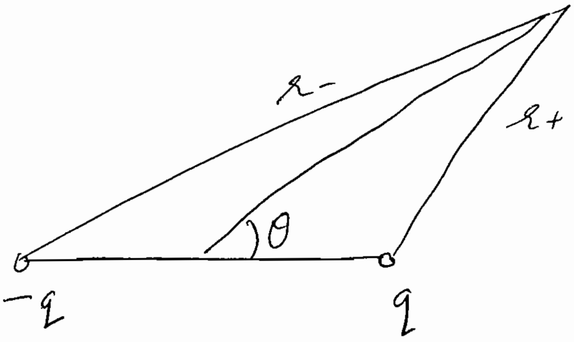
We know, $\frac{d\vec{p}}{d\vec{r}} = 0 \rightarrow \vec{\nabla}(\vec{p}\cdot\vec{r}) = \vec{p}$

$\vec{\nabla}\left(\frac{1}{r^3}\right) = \frac{-3\vec{r}}{r^5}$

⊛ $\nabla(\vec{A}\cdot\vec{r}) = \vec{A}$ for any $\vec{A}$ (const.)

⊛ $\nabla\left(\frac{1}{r^3}\right) = \sum \frac{\partial}{\partial x}\frac{1}{(x^2+y^2+z^2)^{3/2}} = \sum -\frac{3}{2}\cdot(x^2+y^2+z^2)^{-5/2}\cdot 2x = \left(\frac{-3\vec{r}}{r^5}\right)$

$$r_+ = \sqrt{r^2 + \left(\frac{l}{2}\right)^2 - r l \cos\theta}$$



$$= r\sqrt{1 + \left(\frac{l}{2r}\right)^2 - \left(\frac{l}{r}\right)\cos\theta}$$

$$= r\left[1 - \underbrace{\left\{\frac{l}{r}\cos\theta - \left(\frac{l}{2r}\right)^2\right\}}_{\epsilon_1}\right]^{1/2}$$

$$\frac{1}{r_+} = \frac{1}{r}\left[1 - \epsilon_1\right]^{-1/2}$$

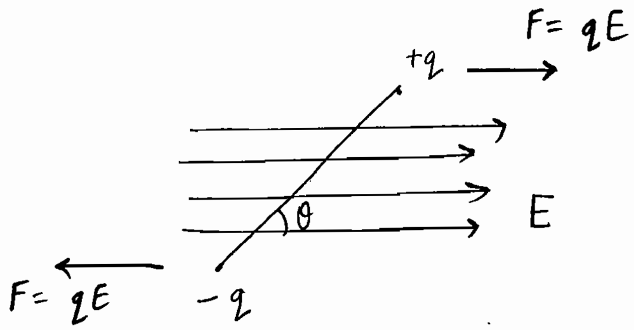
$$r_- = \sqrt{r^2 + \left(\frac{l}{2}\right)^2 + r l \cos\theta}$$

$$= r\left[1 + \underbrace{\left\{\left(\frac{l}{2r}\right)^2 + \left(\frac{l}{r}\cos\theta\right)\right\}}_{\epsilon_2}\right]^{1/2}$$

$$r_- = r\left[1 + \epsilon_2\right]^{+1/2}$$

$$\frac{1}{r_-} = \frac{1}{r}\left[1 + \epsilon_2\right]^{-1/2}$$

$$V(r,\theta) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_+} - \frac{1}{r_-}\right]$$



$F_{net} = 0$

$$\tau = |\vec{r} \times \vec{F}| = qEl\sin\theta = pE\sin\theta$$

$$\vec{\tau} = \vec{p} \times \vec{E}$$

$$dU = +dW_{ext} = +\tau_{ext}\,d\theta = +\left[+pE\sin\theta\,d\theta\right]$$

$$\int U = \int_{\theta_0}^{\theta} \qquad U - U_0 = -pE\cos\theta$$

$$\Rightarrow \boxed{U - U_0 = -pE\cos\theta = -\vec{p}\cdot\vec{E}}$$

✪ इस पे दिमाग नहीं लगाना !!

Take $\tau_{ext} = +pE\sin\theta$
& just mention,
taking $U|_{\theta=90°} = 0$
& do definite integral $\int_{90}^{\theta}$

@ $\theta = 90°$, $U = U_0$ ✓

If the field is non-uniform,

$$\boxed{\vec{F} = (\vec{p}\cdot\vec{\nabla})\,\vec{E}}$$

$$= \left(p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} + p_z \frac{\partial}{\partial z}\right)\cdot\left[E_x \hat{i} + E_y \hat{j} + E_z \hat{k}\right]$$

$$= \left[p_x \frac{\partial E_x}{\partial x} + p_y \frac{\partial E_x}{\partial y} + p_z \frac{\partial E_x}{\partial z}\right]\hat{i}$$

$$\left[p_x \frac{\partial E_y}{\partial x} + p_y \frac{\partial E_y}{\partial y} + p_z \frac{\partial E_y}{\partial z}\right]\hat{j}$$

$$\left[p_x \frac{\partial E_z}{\partial x} + p_y \frac{\partial E_z}{\partial y} + p_z \frac{\partial E_z}{\partial z}\right]\hat{k}$$

[Note that for $\vec{E} = E_0\,\hat{x}$, $\vec{F} = 0$]

$$\boxed{\vec{d} = \vec{r}_{(+)} - \vec{r}_{(-)}}$$

Derivation

$$\vec{F} = -q\vec{E}_- + q\vec{E}_+ = q[\vec{E}_+ - \vec{E}_-] = q\,d\vec{E}$$

$$= q\,[\Delta E_x\,\hat{i} + \Delta E_y\,\hat{j} + \Delta E_z\,\hat{k}]$$

$$= \sum q\,\Delta E_x\,\hat{i} = \sum q.\,\vec{\nabla}E_x\cdot\vec{d}\;\hat{i}$$

$$= \sum q\,\vec{d}\cdot\vec{\nabla}E_x\,\hat{i}$$

$$= \sum \vec{p}\cdot\vec{\nabla}\,E_x\,\hat{i}$$

$$= (\vec{p}\cdot\vec{\nabla})\vec{E}$$

$$\boxed{F = (\vec{p}\cdot\vec{\nabla})\vec{E}}$$

Main point was: $\Delta E_x = \vec{\nabla}E_x\cdot\vec{d}$

$$d E_x = (\vec{E}\cdot\vec{\nabla})\, dx$$
$$d E_y = (\vec{E}\cdot\vec{\nabla})\, dy$$
$$d E_z = (\vec{E}\cdot\vec{\nabla})\, dz$$

$$\Rightarrow \vec{F} = q\,[\,(\vec{E}\cdot\vec{\nabla})\, dx\,\hat{i} + (\vec{E}\cdot\vec{\nabla})\, dy\,\hat{j} + (\vec{E}\cdot\vec{\nabla})\, dz\,\hat{k}\,]$$

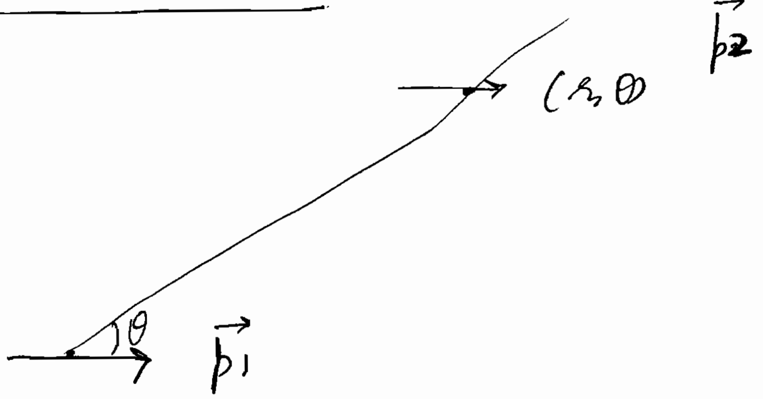
$$\boxed{F = q(\vec{dx}\cdot\vec{\nabla})\vec{E} = (\vec{p}\cdot\vec{\nabla})\vec{E}}$$

Note that whether force is 0 or non-zero τ will act [unless placed parallel to $\vec{E}$]

$\rightarrow \tau = \vec{r}\times\vec{F}$

$\rightarrow U = -\vec{p}\cdot\vec{E}$

<u>Dipole Dipole Interaction</u>



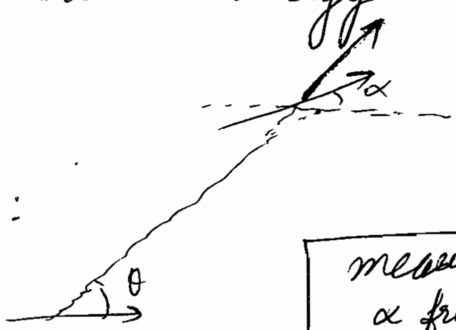
$$U = -\vec{p}_2\cdot\vec{E}_1 = -\vec{p}_1\cdot\vec{E}_2$$

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\vec{p}_1 \cdot \vec{r})\,\vec{r}}{r^5} - \frac{\vec{p}_1}{r^3}\right]$$

$$\boxed{U = \frac{1}{4\pi\epsilon_0}\left[\frac{(\vec{p}_1 \cdot \vec{p}_2)}{r^3} - \frac{3(\vec{p}_1\cdot\vec{r})(\vec{p}_2\cdot\vec{r})}{r^5}\right]}$$

Most stable configuration, i.e. minimum energy

p_1 and p_2 : α angle



measure α from $\hat{r}$

(*) remember from multipole expansion:

(θ) or ($-\theta$) won't matter as dependence is upon $\cos\theta$

$$U = \frac{1}{4\pi\epsilon_0}\left[\frac{p_1 p_2 \cos\alpha}{r^3} - \frac{3\, p_1\cos\theta\, p_2 \cos(\theta-\alpha)}{r^3}\right]$$

$$= \frac{1}{4\pi\epsilon_0 r^3}\, p_1 p_2 \left[\cos\alpha - 3\cos\theta\cos(\theta-\alpha)\right]$$

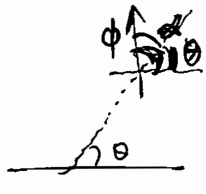
$$= \frac{1}{4\pi\epsilon_0 r^3}\, p_1 p_2 \left[\cos\alpha - \frac{3}{2}\left[\cos(2\theta-\alpha) + \cos(\alpha)\right]\right]$$

$$= \frac{1}{4\pi\epsilon_0 r^3}\, -p_1 p_2 \left[\frac{\cos\alpha}{2} + \frac{3\cos(2\theta-\alpha)}{2}\right]$$

$$\frac{dU}{d\alpha} = 0 \qquad -\sin\alpha + 3\sin(2\theta-\alpha) = 0$$

$$\Rightarrow \sin(2\theta-\alpha) = \frac{1}{3}(\sin\alpha)$$

(*) Note that upon calculation, we can see that α points in the direction of E_1



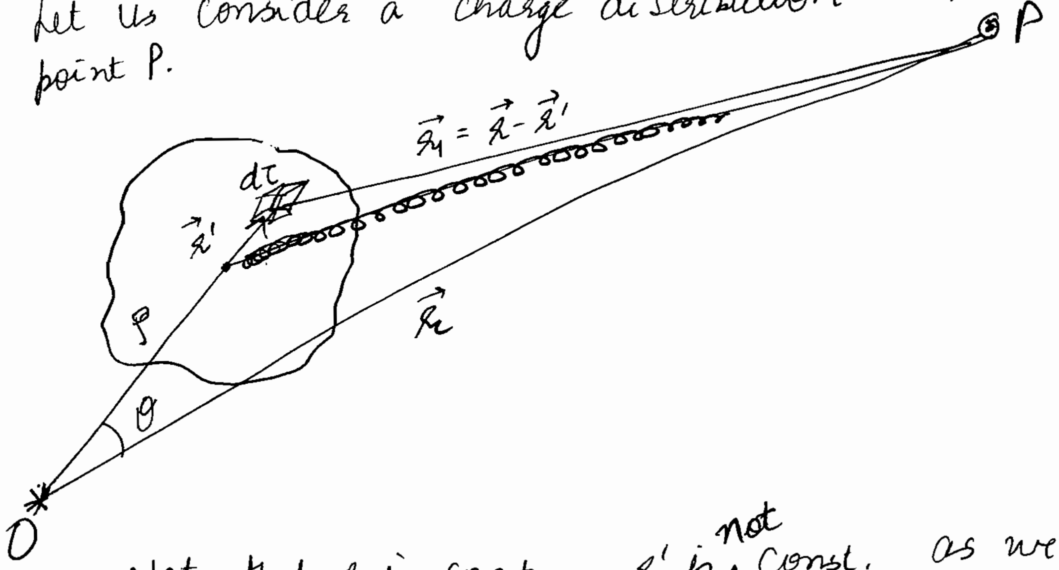
write $\alpha = \theta + \phi$ & $\phi = \tan^{-1}\left(\frac{\sin\theta}{2\cos\theta}\right)$ ✓

E&M (6)

19/1/2012

Multipole Expansion:

→ Let us consider a charge distribution. To find out Potential at point P.



Note that r is const. r' is not Const. as we take different dq.

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r_1}$$

$$V = \int dV = \int \frac{1}{4\pi\epsilon_0}\frac{\rho\, d\tau}{r_1}$$

$$r_1 = \sqrt{r'^2 + r^2 - 2rr'\cos\theta} = r\sqrt{\underbrace{\left\{\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\theta\right\}}_{\epsilon} + 1}$$

$$\frac{1}{r_1} = \frac{1}{r}(1+\epsilon)^{-1/2} \qquad \Rightarrow r_1 = r\sqrt{1+\epsilon}$$

$$\epsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta\right)$$

$$\frac{1}{r_1} = \frac{1}{r}\left[1 - \frac{1}{2}\epsilon + \frac{3\epsilon^2}{8} - \frac{5}{16}\epsilon^3 + \dots\right]$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\left\{\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\theta\right\} + \frac{3}{8}\left\{\left(\frac{r'}{r}\right)^4 + 4\left(\frac{r'}{r}\right)^2\cos^2\theta - 4\left(\frac{r'}{r}\right)^3\cos\theta\right\} - \frac{5}{16}\{\quad\}\right]$$

(पूरा खोल दिया.....
no approximation
is used)

⊛ Only open till $\left(\frac{3}{8}\right)$ term.... do not go for cubic .... till here we will get P_0, P_1 and P_2 ..... We can say that rest follow!!!

$$= \frac{1}{r}\left[1\left(\frac{r'}{r}\right)^0 + \cos\theta\left(\frac{r'}{r}\right)^1 + \left(\frac{r'}{r}\right)^2\left(-\frac{1}{2}+\frac{3}{2}\cos^2\theta\right) + \left(\frac{r'}{r}\right)^3\left(\quad\right)\cdots\right.$$

$$\boxed{\frac{1}{r_1} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta)}$$

✪ Surprisingly the coefficients of $\left(\frac{r'}{r}\right)$ come out as legendre poly.

✪ $\left(\frac{1}{r_1}\right)$ is called generating function of legendre's polynomial.

$$\int dV = \frac{1}{4\pi\epsilon_0}\int \frac{\rho' d\tau'}{r_1} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')}{r}\sum\left(\frac{r'}{r}\right)^n P_n(\cos\theta)\, d\tau$$

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int \rho(r')\,(r')^n\, P_n(\cos\theta)\, d\tau}$$

Multipole Expansion

✪ This equation is exact and no approximation is used to derive it. But primarily its used in approximation scheme: the lowest non-zero term in the expansion provides approximate potential at large r, and the successive terms tell us how to improve the approximation if more precision is required.

$$V(r,\theta)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\int \rho(r')\, d\tau}{r}$$ Monopole $(n=0)$ (Choose charge @ the origin)

$$+ \frac{1}{4\pi\epsilon_0}\frac{\int \rho(r')\, r'\cos\theta\, d\tau}{r^2}$$ Dipole $(n=1)$

$$+ \frac{1}{4\pi\epsilon_0}\frac{\int \rho(r')\, r'^2\,(3\cos^2\theta - 1)\, d\tau}{2r^3}$$ Quadrupole $(n=2)$

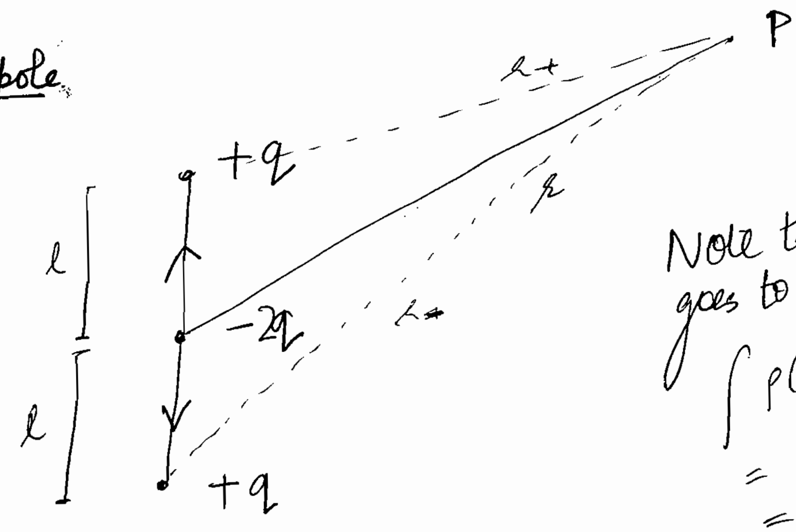
- Hence a charge distribution can behave as Monopole, Dipole, Quadrupole, their combinations.

At $r \gg r'$: Monopole, At $r \simeq r'$: Multipole

At $r > r'$: Dipole

- EM waves are produced when charge particles are accelarated

Linear Quadrupole



Note that dipole term goes to 0 as

$$\int \rho(r')\, r' \cos\theta \, d\tau = ql\cos\theta - ql\cos\theta = 0$$

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0} \frac{ql^2}{r^3}(3\cos^2\theta - 1)$$ ✓

Quadrupole term of multipole expansion will come

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r_+} + \frac{q}{r_-} - \frac{2q}{r}\right]$$

- एक $\left(\frac{1}{r}\right)$ बाहर भी होता है !!

$$= \frac{q}{4\pi\epsilon_0 r}\left[\frac{1}{\sqrt{1+\epsilon_1}} + \frac{1}{\sqrt{1+\epsilon_2}} - \frac{2}{1}\right]$$

- $\left(\frac{r'}{r}\right)^1$ terms are dipole terms

$$= \frac{q}{4\pi\epsilon_0 r}\left[\left(1 - \frac{\epsilon_1}{2} + \frac{3}{8}\epsilon_1^2\right) + \left(1 - \frac{\epsilon_2}{2} + \frac{3}{8}\epsilon_2^2\right) - 2\right]$$

$$r_+ = r\sqrt{1 + \left(\frac{l}{r}\right)^2 - \frac{2l}{r}\cos\theta}$$

they get cancelled on addition hence no dipole term

$$= \frac{q}{4\pi\epsilon_0 r}\left[\frac{3}{4}\frac{l^4}{r^4} + (3\cos^2\theta - 1)\frac{l^2}{r^2}\right]$$

$$r_- = r\sqrt{1 + \left(\frac{l}{r}\right)^2 + \frac{2l}{r}\cos\theta}$$

Now just add $\left(\frac{r'}{r}\right)^2$ वाली terms

$$\approx \frac{q}{4\pi\epsilon_0 r}\left[(3\cos^2\theta - 1)\frac{l^2}{r^2}\right]$$ ✓✓

Dielectrics & Polarization

- Molecules are always in constant motion. i.e. Atoms are in motion; nucleus & electrons are in motion.

- Molecular level dipoles are randomly aligned in absence of $\vec{E}_{ext}$.

- When $\vec{E}_{ext}$ is applied, dipoles align in a particular direction

Every dielectric can be modified.

Quantifiable quantity is Intensity of Polarization $\vec{P}$

→ $\vec{P}$ = (No. of dipoles per unit volume) in the direction of alignment of dipoles.

k: dielectric const. electrical property of matter

ϵ_0 : Permittivity of free space

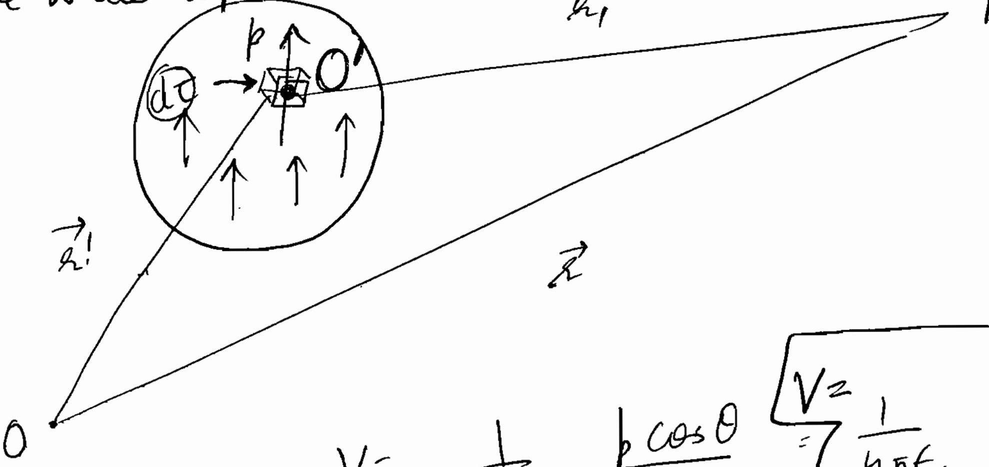
ϵ : Permittivity of a medium

$$\epsilon = \epsilon_0 k$$

$\vec{p} = \vec{P} \cdot d\tau$ → If $d\tau$ is very small s.t. it contains a single dipole.

Since $Q_{total} = 0 \Rightarrow$ we have Monopole term $= 0$

The only term we are interested in is dipole term. So we write dipole term taking the charge elements as origin itself r_1



By taking the origin itself to O', we can write

$$V = \frac{1}{4\pi\epsilon_0} \frac{p\cos\theta}{r_1^2} = \boxed{V = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\vec{r}_1}{r_1^3}}$$

Now we will vary the Origin itself

Due to $d\tau$, $dV = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\vec{r}_1}{r_1^3} = \frac{1}{4\pi\epsilon_0} \frac{\vec{P}\,d\tau\cdot\vec{r}_1}{r_1^3}$

We known $\boxed{\vec{\nabla}\cdot(\phi\vec{A}) = \phi\,\vec{\nabla}\cdot A + \vec{A}\cdot\vec{\nabla}\phi}$

1 scalar और 1 vector को अलग करो !!

$$\boxed{\nabla\cdot\left(\frac{1}{r_1}\vec{P}\right) = \frac{1}{r_1}\nabla\cdot\vec{P} + \vec{P}\cdot\nabla\left(\frac{1}{r_1}\right)} \quad \text{—— (1)}$$

Now we can write

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\int \vec{P}\,d\tau\cdot\nabla\left(\frac{1}{r_1}\right)$$

Note that $r = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$

Grad w.r.t. final point $\nabla\left(\frac{1}{r}\right) = \left(-\frac{\vec{r}}{r^3}\right)$ (final point is variable (x',y',z') : fixed)

Grad. w.r.t. source point $\nabla\left(\frac{1}{r}\right) = \left(\frac{\vec{r}}{r^3}\right)$ (initial point is variable (x,y,z) fixed ⇒ one (−) sign of differentiation in numerator)

Now using (1),

$$V(r,\theta) = \frac{-1}{4\pi\epsilon_0}\int_V \frac{1}{r_1}\nabla\cdot\vec{P} - \nabla\cdot\left(\frac{\vec{P}}{r_1}\right) d\tau$$

$$= \frac{-1}{4\pi\epsilon_0}\int_V \frac{1}{r_1}(\vec{\nabla}\cdot\vec{P})\,d\tau + \frac{1}{4\pi\epsilon_0}\int_V +\left(\vec{\nabla}\cdot\frac{\vec{P}}{r_1}\right)d\tau$$

From reverse of divergence theorem

$$= \frac{+1}{4\pi\epsilon_0}\int\frac{-(\vec{\nabla}\cdot\vec{P})}{r_1}\,d\tau + \frac{1}{4\pi\epsilon_0}\oint_S\frac{+\vec{P}\cdot\vec{ds}}{r_1}$$

Hence $-\vec{\nabla}\cdot\vec{P} = \rho_b$ [defining] Bound Volume Charge Density

$+(\vec{P}\cdot\hat{n}) = \sigma_b$ Bound Surface Charge Density

$$\boxed{V(r,\theta) = \frac{1}{4\pi\epsilon_0}\int_V \frac{(\rho_b \, d\tau)}{r_1} + \frac{1}{4\pi\epsilon_0}\oint_S \frac{\sigma_b \, ds}{r_1}}$$

where $\rho_b = -\vec{\nabla}\cdot\vec{P}$

$\sigma_b = \vec{P}\cdot\hat{n}$

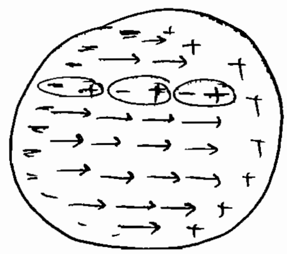
If matter is uniformly polarized $\Rightarrow$ P is same everywhere

$\Rightarrow \vec{\nabla}\cdot\vec{P} = 0$

Hence ρ_b arises only when non-uniform Polarization.

Note that the charge here is not conduction charge like current. We cannot light bulb from this charge flow. ($\because$ its fixed charge distribution & not a flowing charge)

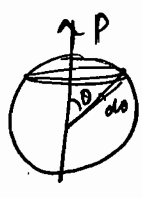
Surface charge density will be there whenever Polarization occurs.



Gas & liquids are always uniformly polarized, hence $-\nabla\cdot\vec{P} = \rho_b = 0$ for fluids.

$$q_b = \int_V \rho_b \, d\tau + \int_S \sigma_b \, ds$$

Note that $\oint \sigma_b \, ds = 0$ ......



$$q_s = \oint \sigma \cdot ds = \int_0^{\pi} P\cos\theta \cdot 2\pi R^2 \sin\theta \, d\theta = 0$$

Q29)

Do not desire ... -42

Just use $-\vec{\nabla}\cdot\vec{P}$

Write 2-3 lines of Physics

Polarizability :-

Dipole [redacted] Moment of a molecules due to field experienced by a molecule.

$$\vec{p}_m = \alpha \vec{E}_0$$

$\vec{E}_0$ ← applied field

α ← Constant of Proportionality

α is due to

- valency electrons (α_e)
- ionic polarizability (α_{io})

$$A \longrightarrow A^+ + e^-$$

A^+ ← (displaced along the field)

e^- ← (displaced opposite to the field.)

- Orientation Polarizability ($\alpha_{orientation}$)

→ atomic polarizability = [electronic + ionic + Orientation] polarizabilities

hence, $\alpha = [\alpha_{elect} + \alpha_{ionic} + \alpha_{orientation}]$

$$\vec{p}_m = \alpha \vec{E}_0 \checkmark$$

Field due to Uniform Polarization :-

note that this derived for general dielectric (sphere) and not just linear

✓ $$\boxed{\vec{E}_{in} = \vec{E}_{else} - \frac{\vec{P}}{3\epsilon_0}}$$ inside the dielectric

✓ It behaves as a dipole with $|\vec{p}| = |\vec{P}| * \text{Volume}$, $\vec{p} = (4/3)\pi R^3 \vec{P}$ outside the [redacted] dielectric

$$\vec{E}_{Polarization} = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma_b \, ds}{r^3} \vec{r}$$

$$= -\left(\frac{P}{3\epsilon_0}\right)$$

Clausius - Mossottee Relation

Relates 'α' with no. of dipoles per unit volume.
for fluids.

Let there be n dipoles per unit volume of dielectric medium

$$\Rightarrow \quad \vec{P} = n \, p_m = n\alpha \vec{E}_{ext}$$

$$= n\alpha \left[\vec{E}_{total} + \frac{\vec{P}}{3\epsilon_0} \right]$$

$$\Rightarrow \boxed{\vec{P}\left[1 - \frac{n\alpha}{3\epsilon_0}\right] = n\alpha \vec{E}_{total}} \quad \text{—— (1)}$$

$\Rightarrow$

We know from Gauss law,

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \vec{E} = \frac{\rho_{conduction} + \rho_{bound}}{\epsilon_0} = \left(\frac{\rho_{bound} + \rho_{free}}{\epsilon_0}\right)$$

$$\Rightarrow \nabla\cdot(\epsilon_0 \vec{E}) = -(\vec{\nabla}\cdot\vec{P}) + \rho_{free}$$

$$\Rightarrow \nabla\cdot(\epsilon_0\vec{E}+\vec{P}) = \rho_{free}$$

Let $\boxed{\vec{D} = \epsilon_0 \vec{E} + \vec{P}}$

$$\Rightarrow \boxed{\nabla\cdot\vec{D} = \rho_{free}}$$

Gauss law in dielectric medium

For air, $P = 0$

$$\vec{D} = \epsilon_0 \vec{E}$$

For any dielectric

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon\vec{E} = k\,\epsilon_0\vec{E}$$

Putting in (1)

$$\frac{|\vec{P}|}{|\vec{E}_0|} = (k-1)\,\epsilon_0$$

Let $\dfrac{P}{\epsilon_0 E} = \chi$: dielectric susceptibility

$$\boxed{1+\chi = k}$$

$$\Rightarrow \quad 1 - \frac{n\alpha}{3\epsilon_0} = \frac{n\alpha}{(k-1)\epsilon_0}$$

$$\Rightarrow \quad 1 = n\alpha\left[\frac{1}{(k-1)\epsilon_0} + \frac{1}{3\epsilon_0}\right] = \frac{n\alpha}{3\epsilon_0}\left[\frac{3}{k-1} + 1\right]$$

$$\Rightarrow \quad 1 = \frac{n\alpha}{3\epsilon_0}\left(\frac{k+2}{k-1}\right)$$

$$\Rightarrow \quad \boxed{\frac{(k-1)}{(k+2)} = \frac{n\alpha}{3\epsilon_0}}$$

Classius Mossottee Relationship

Microscopic property

macroscopic property

$$\alpha = \frac{3}{4\pi N}\left(\frac{\epsilon - 1}{\epsilon - 2}\right)$$

Derivation of Classius Mossottee

Let a dielectric Object kept in field E_{ext}.

Now $\vec{P} =$ [redacted] $n\alpha \vec{E}_{ext}$ — (1)

Also for linear dielectric $\vec{P} = \chi_e \epsilon_0 \vec{E}_{total}$ — (2)

& $\vec{E}_{Total} = \vec{E}_{ext} - \frac{\vec{P}}{3\epsilon_0}$ — (3)

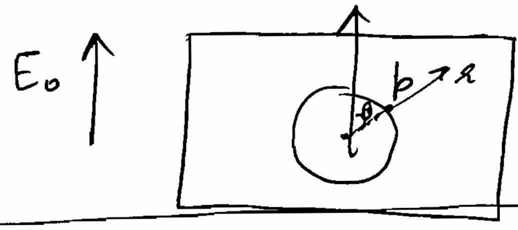
$$\Rightarrow \quad \frac{\vec{P}}{\chi_e \epsilon_0} = \frac{\vec{P}}{n\alpha} - \frac{\vec{P}}{3\epsilon_0}$$

$$\Rightarrow \quad \frac{1}{\chi_e \epsilon_0} + \frac{1}{3\epsilon_0} = \frac{1}{n\alpha}$$

$$\Rightarrow \quad \frac{n\alpha}{3\epsilon_0} = \frac{\chi_e \epsilon_0}{3\epsilon_0 + \chi_e \epsilon_0} = \frac{\chi_e}{3 + \chi_e} = \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)$$

E & M (7)

20/01/2012



$$\rho_b = \vec{\nabla}\cdot\vec{P} = 0$$

$$\sigma_b = \vec{P}\cdot\hat{r} = P\cos\theta$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma_b\, ds}{r^2}\,\hat{r}$$

$$ds = 2\pi r\sin\theta\; r\, d\theta$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_0^{\pi} \frac{(P\cos\theta)\cdot\cos\theta\; ds}{r^2}$$

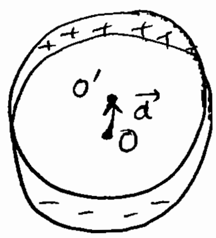
$$= \frac{P}{4\pi\epsilon_0}\int_0^{\pi} \frac{\cos^2\theta\cdot 2\pi r^2\sin\theta\, d\theta}{r^2}$$

$$= \frac{P}{2\epsilon_0}\int_0^{\pi} \cos^2\theta\,\sin\theta\, d\theta$$

$$= \frac{P}{2\epsilon_0}\cdot\frac{2}{3} = \frac{\vec{P}}{3\epsilon_0}$$

$$\boxed{\vec{E} = \frac{\vec{P}}{3\epsilon_0}}$$

To analyze uniformly polarized sphere, we can take the following analogy:



2 spheres of charge (Opposite) q and $-q$ are overlapped but displaced slightly. The leftover charge at top & bottom are equivalent to Polarization surface charge density.

$\Rightarrow$ At any point in region of overlap, we can calculate from Gauss law

$$\vec{E} = -\frac{1}{4\pi\epsilon_0}\cdot\frac{q\vec{d}}{R^3}$$

We can express it in terms of Polarization of sphere as $\vec{p} = q\vec{d} = \left(\frac{4}{3}\pi R^3\right)\vec{P}$

$$\Rightarrow \vec{E} = -\left(\frac{P}{3\epsilon_0}\right)$$

For outside points, $V = \frac{1}{4\pi\epsilon_0}\left(\frac{\vec{p}\cdot\hat{r}}{r^2}\right)$

Electric Field in Matter

(*) On a broad scale, materials are divided into two types: Conductors and Insulators (or dielectrics)

Conductors are substances that have "unlimited" supply of charges the are free to move.

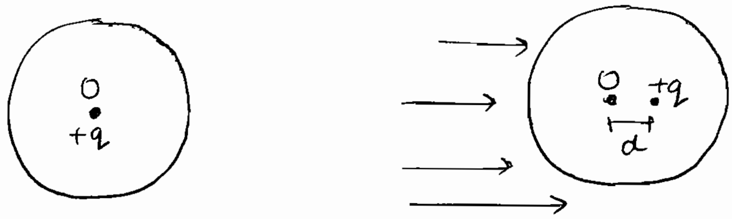
In dielectrics, in contrast, all charges are attached to specific atom or molecule – they are not free to move but can move a bit within the atom or molecule. Such microscopic displacements are not as dramatic as in a conductor, but their cumulative effects account for characteristic behaviour of dielectrics.

(*) When placed in an Electric Field (weak), a neutral atom acquires a tiny dipole moment $\vec{p}$ which points in the same direction as $\vec{E}$. (Note that if the field is large enough, it can pull the atom apart completely, "ionizing" it and the dielectric becomes a conductor)

$$\vec{p} = \alpha \vec{E}_{applied}$$ α: atomic polarizability

Q/ Consider a model of atom (volume = v) of a point nucleus ($+q$) surrounding by a uniformly spherical cloud ($-q$) of radius a. Calculate atomic polarizability of such an atom.

Ans/



In presence of electric field $\vec{E}$, the nucleus will be shifted towards right and the electron cloud to the left. Assume e-cloud maintains spherical shape.

Now at equilibrium, Force upon nucleus balance.

$$\Rightarrow Eq = \frac{1}{4\pi d^2}\cdot\frac{q^2}{\varepsilon_0 a^3}\, d^3 \Rightarrow \frac{qd}{4\pi\varepsilon_0 a^3} = E \Rightarrow p = 4\pi\varepsilon_0 a^3 E$$

$$\Rightarrow \boxed{\frac{p}{E} = 3v\varepsilon_0}$$

(*) If the material (dielectric) already has dipole moment, then the tiny dipoles experience a torque and align along electric field.

∴ the two mechanism (induced dipole moment and torque) produce the same basic result : a lot of dipoles pointing along the direction of the field - the material becomes polarized. A convenient measure of this effect is $\underline{P}$, the dipole moment per unit volume. Its called POLARIZATION.

(*) Solving for Potential, we get equivalent of dielectric of Polarization P is

surface charge of $\vec{P}\cdot\hat{n}$

volume charge of $-\vec{\nabla}\cdot\vec{P}$

Q/ A specified charge density $\sigma_0(\theta)$ is glued over surface of a spherical shell of radius R. Find the resulting potential inside and outside of sphere.

जाभा ही सोखा हैं !!

A/ Interior

$$V(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta)$$

So that $V(r,\theta)$ does not blow up at $r=0$, we have $B_n = 0$

$$\Rightarrow V(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

Outside

$$V(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta)$$

So that $V(r,\theta)$ goes to 0 at ∞, we have $A_n = 0$ $(n>0)$

$$\Rightarrow V(r,\theta) = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta)$$

① The Potential must be continuous at $r = R$

$$\sum_{n=0}^{\infty} A_n R^n P_n(\cos\theta) = \sum_{n=0}^{\infty} B_n R^{-(n+1)} P_n(\cos\theta)$$

Comparing coefficients of $P_n(\cos\theta)$

$$\boxed{B_n = A_n R^{2n+1}}$$

② The radial derivative of V suffers a discontinuity at the surface

(From gauss law, $E_{above} - E_{bottom} = \frac{\sigma_f}{\varepsilon_0}\hat{n}$) ★

do not forget this minus sign

$$\Rightarrow \left.\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right|_{r=R} = -\frac{1}{\varepsilon_0}\sigma_0(\theta)$$

$$\Rightarrow -\sum_{n=0}^{\infty}(n+1)\frac{B_n}{R^{n+2}}P_n(\cos\theta) - \sum_{n=0}^{\infty} n A_n R^{n-1} P_n(\cos\theta) = -\left(\frac{\sigma_0(\theta)}{\varepsilon_0}\right)$$

$$\sum (2n+1) A_n R^{n-1} P_n(\cos\theta) = \frac{\sigma_0(\theta)}{\varepsilon_0}$$

using Fourier not reqd. $\Rightarrow$

$$\boxed{A_n = \frac{1}{2\varepsilon_0 R^{n-1}}\int_0^{\pi} \sigma_0(\theta)\, P_n(\cos\theta)\sin\theta\, d\theta}$$

Specific Case

$\sigma_0(\theta) = k\cos\theta$

All A_n are zero except $n = 1$

$\Rightarrow 3A_n\cos\theta = \frac{k\cos\theta}{\varepsilon_0}$

$\Rightarrow \boxed{A_1 = \left(\frac{k}{3\varepsilon_0}\right)}$

$$\Rightarrow A_1 = \frac{k}{2\varepsilon_0}\int_0^{\pi}\cos^2\theta\,\sin\theta\, d\theta = \frac{k}{2\varepsilon_0}\cdot\left(\frac{2}{3}\right)$$

$$= \left(\frac{k}{3\varepsilon_0}\right)$$

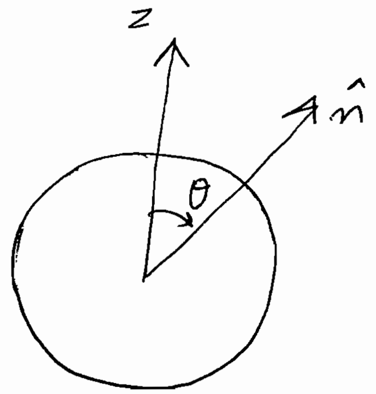
$$\Rightarrow B_1 = \left(\frac{k R^3}{3\varepsilon_0}\right)$$

$$\Rightarrow V_{inside} = \frac{k}{3\varepsilon_0} r \cos\theta \qquad (r \leq R)$$

$$V_{outside} = \frac{k}{3\varepsilon_0} \cdot \frac{R^3}{r^2} \cos\theta \qquad (r \geq R)$$

Field due to uniformly polarized sphere of radius R

Let us choose $\hat{z}$ as the direction of $\vec{P}$



Now $\rho_b = 0$ (uniform $\vec{P}$) $\uparrow \vec{P}$

$\sigma_b = \vec{P} \cdot \hat{n} = P\cos\theta$

Now the potential due to σ_b, is given as

$$\boxed{V_{inside} = \frac{P}{3\varepsilon_0} \cdot r\cos\theta} \qquad (r \leq R)$$

$$\boxed{V_{outside} = \frac{P}{3\varepsilon_0} \cdot \frac{R^3}{r^2} \cos\theta} \qquad (r \geq R)$$

[Note that these results can be directly lifted as the uniform Polarization ensures that laplace equation is valid both inside and outside the sphere]

$$\vec{E}_{in} = -\vec{\nabla} V_{in} = -\frac{P}{3\varepsilon_0}\hat{z} = \underline{\underline{-\frac{1}{3\varepsilon_0} \cdot \vec{P}}} \qquad r < R$$

$$\vec{E}_{out} = -\vec{\nabla} V_{out} = \frac{P R^3}{3\varepsilon_0} \frac{2}{r^3} \cos\theta \,(\hat{r}) + \frac{P R^3}{3\varepsilon_0 r^2} \sin\theta\, \hat{\theta}$$

$$= \frac{P R^3}{3\varepsilon_0 r^3} \cdot \left(2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta} \right)$$

Now using $p_{total} = P \cdot \frac{4}{3}\pi R^3$

$$= \underline{\underline{\frac{p_{total}}{4\pi\varepsilon_0} \frac{1}{r^3} \left(2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta} \right)}} \qquad r > R$$

$\therefore$ outside the sphere, it acts as dipole of dipolemoment $p_{total} = P * V$

(★) Note that for most dielectrics, $\nabla \cdot P = 0 \Rightarrow \rho_b = 0$

Hence $\int \sigma_b \cdot d\vec{a} = 0$ ∵ the overall dielectric is neutral.

But in the case of non uniform Polarization, ρ_b is not zero and neither is charge due to σ_b

i.e. $Q_{surface} + Q_{volume} = 0$

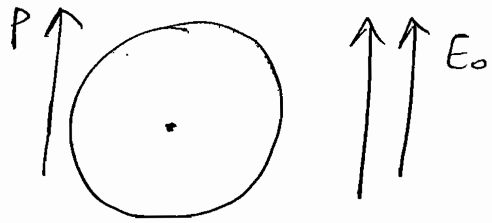
but individually they may not be zero.

Also, $Q_{vol} = \int_V \rho_b \, d\tau = -\int_V \vec{\nabla} \cdot \vec{P} \, d\tau = -\oint \vec{P} \cdot d\vec{a}$

$= -\oint \sigma_b \, ds = -Q_{surface}$

$\Rightarrow \boxed{Q_{volume} + Q_{surface} = 0}$

Field inside dielectric in presence of $\vec{E}$



$E_{inside} = \hat{E}_{outside} + \hat{E}_{polarization}$

$= \left(E_0 - \frac{P}{3\varepsilon_0}\right) \hat{z}$

(Note that $\vec{E}$ obeys superposition principle always.... even for conductor $E_{in} = E_{applied} + E_{conductor, charge} = 0$)

Gauss law for dielectrics

From Gauss law

$\vec{\nabla} \cdot \vec{E} = \left(\frac{\rho}{\varepsilon_0}\right)$

Now in a dielectric ρ may be of two types: bound charge ($\rho_b = -\vec{\nabla} \cdot \vec{P}$) and other free charges that might be present. Lets call them ρ_f. Free charge may consist of electrons in a conductor that may be present in dielectric medium or ions embedded in dielectric medium. or whatever charge that is not a result of polarization.

$$\Rightarrow \boxed{\nabla \cdot E = \frac{\rho}{\varepsilon_0}}$$

Note that Gauss law remains same but it can be expressed in a different form for dielectric medium

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho_b + \rho_f}{\varepsilon_0} = \frac{-\nabla\cdot P + \rho_f}{\varepsilon_0}$$

$$\Rightarrow \varepsilon_0 \vec{\nabla}\cdot \vec{E} + \vec{\nabla}\cdot\vec{P} = \rho_f$$

$$\Rightarrow \vec{\nabla}\cdot(\varepsilon_0\vec{E} + \vec{P}) = \rho_f$$

Define Electric Displacement Vector $\vec{D}$ as

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

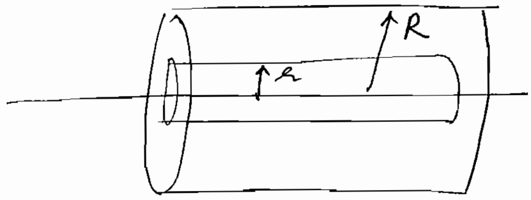
$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{D} = \rho_f}$$ Alternate 'form' of Gauss law for dielectric medium.

In integral form, $\boxed{\oint \vec{D}\cdot d\vec{a} = Q_{\text{free enclosed}}}$

In typical situations, we usually know ρ_{free} but not ρ_{bound}, $\therefore$ this form of Gauss law is particularly useful for dielectrics.

Q/ A long straight wire carrying uniform line charge λ, is surrounded by rubber insulation out to a radius R. Find electric displacement.

A/ From gauss law



$$D \cdot 2\pi r \cdot l = \lambda l$$

$$\boxed{\vec{D} = \frac{\lambda}{2\pi r}\hat{s}}$$

Now note that this formula holds both within and outside the insulation. In the latter region, $\vec{P} = 0$

$$\Rightarrow \quad \vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E}$$

$$\Rightarrow \quad \vec{E}_{outside} = \frac{\vec{D}}{\varepsilon_0} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{s}$$

Inside, the electric field cannot be determined, since we do not know $\vec{P}$.

Some Properties of $\vec{D}$

(1) No Coulombs law for $\vec{D}$

(2) Curl of $\vec{D} \neq 0$

$$\nabla \times \vec{D} = \epsilon_0(\vec{\nabla} \times \vec{E}) + \vec{\nabla} \times \vec{P} = 0 + \vec{\nabla} \times \vec{P} = (\vec{\nabla} \times \vec{P}) \neq 0$$

(3) There is no Potential corresponding to $\vec{D}$.

(4) Boundary Condition

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_{free}$$ (from integral form of Gauss Law for dielectrics) (We can get from def^n also σ_b =

$$D^{\parallel}_{above} - D^{\parallel}_{below} = P^{\parallel}_{above} - P^{\parallel}_{below}$$ (from def^n of $\vec{D}$)

LINEAR DIELECTRICS

For linear dielectrics,

$$\vec{P} \propto \vec{E}$$

$$\Rightarrow \quad \boxed{\vec{P} = \epsilon_0 \chi_e \vec{E}}$$

χ_e : electric susceptibility (dimensionless quantity)

Note that $\vec{E}$ is the total field, it may be due to [redacted]

① free charges or
② Polarization itself

If for instance, we put a piece of dielectric into an external field $\vec{E_0}$, we cannot compute $\vec{P}$ directly from this equation as the external field will polarize the material, and this polarization will produce its own field, which will contribute to total field ... which in turn modifies Polarization and so on till an equilibrium E is achieved. The simplest approach is to begin with displacement $\vec{D}$, at least in those cases where $\vec{D}$ can be deduced directly from free charge dist^n.

In linear media,

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon_0\vec{E} + \chi_e\epsilon_0\vec{E} = \epsilon_0(1+\chi_e)\vec{E}$$

define permittivity of material ϵ, s.t.

$$\epsilon = \epsilon_0(1+\chi_e)$$

$$\Rightarrow \boxed{\vec{D} = \epsilon\vec{E}}$$

define relative permittivity or dielectric constant of the medium, ϵ_r

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = (1+\chi_e)$$

(*) Note that while Polarization is usually in the direction of applied external field while the electric field due to Polarization is opposite to $\vec{P}$ i.e. usually opposite to applied field. Therefore, like a conductor, a dielectric tries to minimize field inside it as much as it can, but the cancellation is partial (unlike conductors where cancellation is complete and $\vec{E}_{inside} = 0$). Dielectric is therefore, a poor conductor.

We have calculated field for a dielectric as

$$\vec{E}_{inside} = \vec{E}_0 - \frac{\vec{P}}{3\epsilon_0}$$

E_0 : E applied
E_{inside} : E_{total}

For linear dielectric, $\vec{P} = \epsilon_0 \chi_e \vec{E}_{total}$

$$\Rightarrow \vec{E} = \vec{E}_0 - \epsilon_0 \chi_e E/3\epsilon_0$$

$$\Rightarrow \vec{E}\left(1 + \frac{\epsilon_0 \chi_e}{3\epsilon_0}\right) = \vec{E}_0$$

$$\Rightarrow \vec{E} = \frac{\vec{E}_0}{\left(1 + \frac{\epsilon_0 \chi_e}{3\epsilon_0}\right)} = \frac{\vec{E}_0}{1 + \frac{\chi_e}{3}} = \left(\frac{3\vec{E}_0}{\chi_e + 3}\right) = \left(\frac{3\vec{E}_0}{\epsilon_r + 2}\right)$$

$$\boxed{\vec{E} = \left(\frac{3}{\epsilon_r + 2}\right)\vec{E}_0}$$

Also, for a linear dielectric,

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$
$$= -\vec{\nabla}\cdot\left(\epsilon_0 \chi_e \vec{E}\right)$$
$$= -\vec{\nabla}\cdot\left(\epsilon_0 \chi_e \frac{\vec{D}}{\epsilon}\right)$$
$$= -\vec{\nabla}\cdot\left(\frac{\epsilon_0 \chi_e}{(1+\chi)\epsilon_0}\vec{D}\right)$$
$$= -\frac{\chi_e}{1+\chi_e}\,\vec{\nabla}\cdot\vec{D}$$
$$= -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_{free}$$

$$\boxed{\rho_b = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_{free}}$$ for linear dielectric

Boundary Conditions for linear dielectrics

We know,

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_{free}$$

$$\Rightarrow \boxed{\epsilon_{above} E^{\perp}_{above} - \epsilon_{below} E^{\perp}_{below} = \sigma_{free}}$$

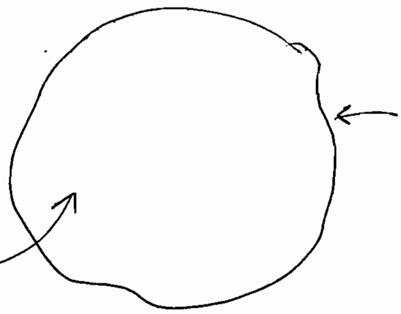
$$\Rightarrow \epsilon_{above} \left(\frac{\partial V_{above}}{\partial n}\right) - \epsilon_{below} \left(\frac{\partial V_{below}}{\partial n}\right) = -\sigma_f$$

Also $\boxed{V_{above} = V_{below}}$ at surface only

Method of Electrical Images

Not all field problems can be solved easily with method of electrical images.

This method is based upon 'Uniqueness Theorem in Electrostatics' i.e for a particular boundary conditions, the solution of laplace & poisson equation is unique. i.e. one and only one.



V_1

$V_{boundary} = V_0$

[it can be taken as 0 with suitable choice of reference]

Now for given V_0, V_1 will be unique!!!

Proof by Contradiction

$$\nabla^2 V = 0$$

Let there be 2 solutions in the region under consideration whose boundary is at Const. Potential (say 0)

The solution of differential equations gives the average values of the functions. They do not tolerate local maxima or local minima. Extreme values of functions lie on the boundary.

Now, Maxima of $V_1 = 0$
Minima of $V_1 = 0$
$\Rightarrow V_1 = 0$

Similarly, Maxima of $V_2 = 0$
Minima of $V_2 = 0$
$\Rightarrow V_2 = 0$

$\Rightarrow V_1 = V_2$

If V_1 and V_2 are solutions, hence linear combination of V_1 and V_2 are also solutions.

$$\Rightarrow \alpha_1 V_1 + \alpha_2 V_2 \quad \text{are solutions of } \nabla^2 V = 0$$

$$\nabla^2 (\alpha_1 V_1 + \alpha_2 V_2) = \alpha_1 \nabla^2 V_1 + \alpha_2 \nabla^2 V_2 = 0 + 0 = 0$$

Hence $V_1 - V_2$ will be a solution,

$$\nabla^2 (V_2 - V_1) = \nabla^2 V_2 - \nabla^2 V_1 = 0 - 0 = 0$$

Its maxima & minima (of $V_1 - V_2$) will lie on the boundary. It will not tolerate local minima & local maxima.

Maxima of $(V_1 - V_2) = 0$

Minima of $(V_1 - V_2) = 0$

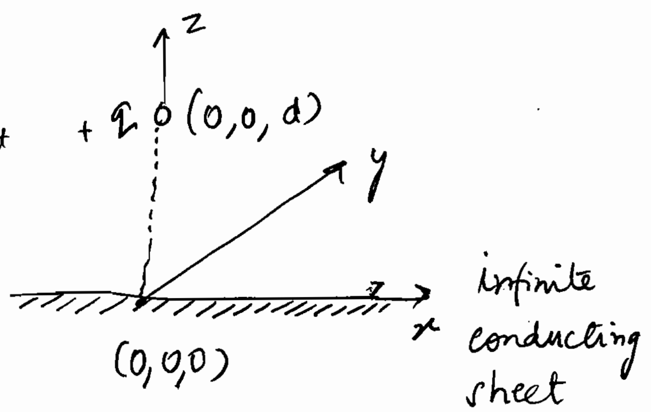
$\Rightarrow V_1 - V_2 = 0$

$\Rightarrow V_1 = V_2$

✓ Note that apart from $\vec{E}, \vec{F}, U, V$, σ can also be calculated from the 5 methods.

There are only two standard problems:

① Infinitely Conducting Sheet

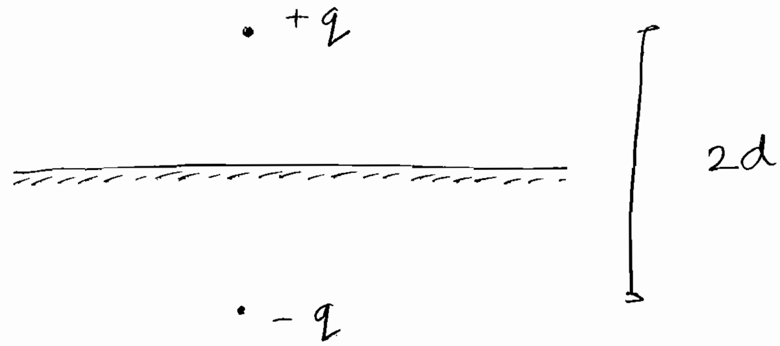


At $z=0$, Potential is always constant

i.e. $V_{z=0} = 0$

Now, according to method of electrical images,

the given problem is equivalent to



Note that main difficulty is to replicate the same boundary conditions

$$F = \frac{1}{4\pi\epsilon_0} \cdot \left(\frac{q^2}{4d^2}\right)$$

Force between 'sheet & charge' is attractive.

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right] \quad - \text{(I)}$$

$$V(x,y,0) = 0 - 0 = 0$$

[Boundary Conditions]

$$dW = \frac{1}{2}\int F\,dz$$

⟸ Note that work is half as the other charge is just virtual

From (I)

$$\vec{E} = -\vec{\nabla} V$$

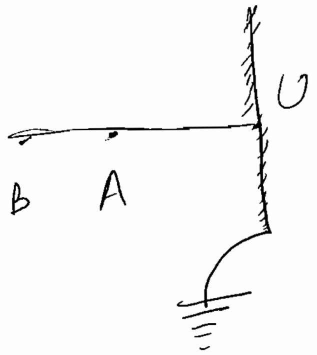
From $E(x,y,z)$ we can find $E(x,y,0)$

$$E(x,y) = \frac{\sigma}{\epsilon_0}$$

$$\boxed{\sigma = \epsilon_0 E(x,y)}$$

$$Q = \int_{-\infty}^{\infty}\int \sigma \, dx \, dy$$

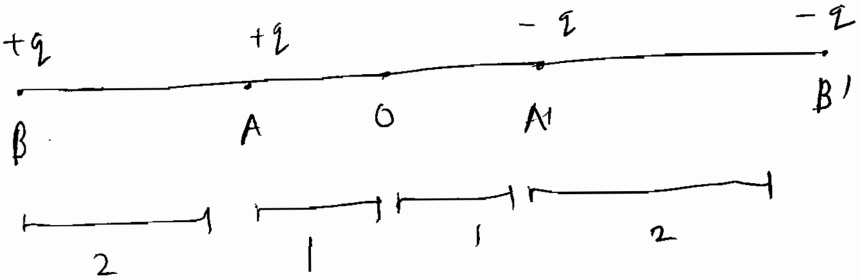
(Q1)



$OA = 1m \quad AB = 2m$

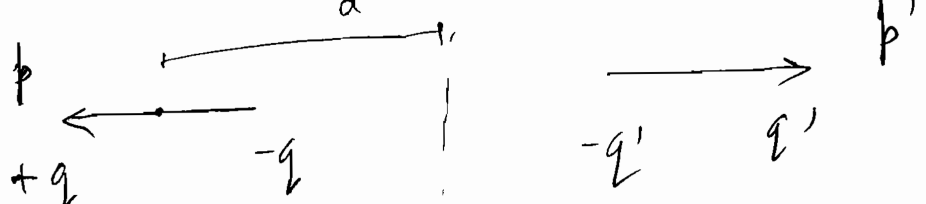
$q_A = q_B = 2 \times 10^{-3}\ C$

Now the problem is equivalent to.



(10)

For dipole,



$$U = \frac{1}{4\pi\epsilon_0}\left[\frac{\vec{p_1}\cdot\vec{p_2}}{r^3} - \frac{3(p_1\cdot r)(p_2\cdot r)}{r^3}\right] \quad \boxed{r = 2d}$$

(*) There are two points worth noting in Laplace equation : $\nabla^2 V = 0$

(1) $V(x)$ is the average of $V(x+a)$ and $V(x-a)$ i.e. laplace equation is a kind of averaging instruction, it tells us to assign to the point x, the average of the values to the left and right of x.

(2) Laplace equation tolerates no local maxima or minima, extreme values of V must be at the end points.

< These validations are for any dimension (x), (x,y) or (x,y,z) >

(*) Solutions of laplace equations are called Harmonic functions.

(*) 

This cannot be potential function as it has a local minima.

Laplace equation will pick the most featureless function possible. eg. if you put a ping-pong ball on a stretched rubber sheet (following the potential function), it will roll over to one side and fall off - it will not find a "pocket" somewhere to settle into, for Laplace equation allows no such dents. eg. just as a straight line is the shortest distance between two points, so a harmonic function in 2-d minimizes the surface area spanning the given boundary line.

(Note that we are talking of Laplace solutions and not Poisson's)

From these observation, ~~they~~ we get EARNSHAW's THEOREM:

[A charged particle cannot be held in a stable equilibrium by electrostatic field alone.]

This is because stable equilibrium requires a valley type of Potential function.

$\therefore$ electrostatic forces cannot balance nuclear fusion in Tokamac.

Fortunately its possible to Confine plasma magnetically.

Uniqueness Theorem

Laplace equation requires a suitable set of boundary conditions to determine V. But what are suitable boundary conditions, sufficient to determine the potential and yet not so strong as to generate inconsistencies.

If 2- or 3 dimensions partial equations are given, its not easy to ~~say~~ what would constitute acceptable boundary conditions. The proof that a proposed set of boundary conditions will suffice is usually presented in the form of a uniqueness theorem. There are two basic uniqueness theorems.

(1) The solution to Laplace's equation in some volume τ is uniquely determined if potential V is specified on the boundary surface of τ.

Proof: let two solutions V_1 and V_2 satisfying

$$\nabla^2 V_1 = 0 \quad , \quad \nabla^2 V_2 = 0$$

Consider $V_3 = V_1 - V_2$

$$\Rightarrow \nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0 - 0 = 0 \quad \text{(laplace eqn)}$$

Now at boundary V_1 and V_2 are equal and specified

$$\Rightarrow V_3 = 0 \quad \text{at boundary surface}$$

But laplace equation allows no local maxima or minima - all extrema occur at the boundary so the maxima & minima of V_3 are both 0

$\Rightarrow$ V_3 is 0 everywhere

$$\Rightarrow \boxed{V_1 = V_2}$$

QED !!

Application

Potential inside an enclosure, having no charges, and completely surrounded by conducting material: Potential on cavity wall is same V_0

One potential function that satisfies laplace equation and has constant value V_0 at the boundary is

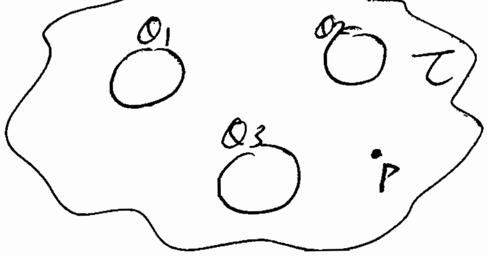
$$V = V_0$$

The uniqueness theorem guarantees, that this is the only solution.

So imagine a solution that satisfies (a) Laplace equation (b) has correct value at boundaries

This is the right solution.

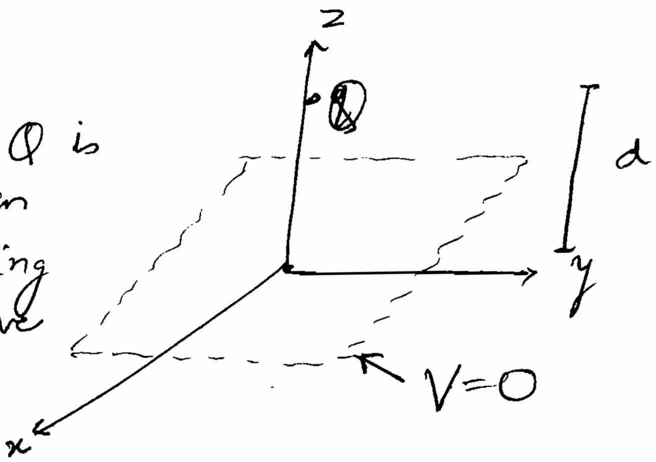
(2) In a volume τ surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if total charge on each is specified (conductor)



Method of Images

(1) Suppose a point charge Q is held a distance d above an infinite grounded conducting plane. For the region above the plane, V is not just

$\frac{1}{4\pi\varepsilon}\left(\frac{q}{\cancel{R}}\right)$ $\because$ q is induce



some negative charge in conductor which we don't know is how much or how distributed.

From a mathematical point of view, our problem is to solve Poisson's equation in region $z > 0$, with a single point charge q at $(0,0,d)$, subject to boundary condition:

(i) $V = 0$ when $z = 0$
(ii) $V \to 0$ where $r \to \infty$

Corollary to 1st Uniqueness theorem

If there is some charge density ρ

$$\nabla^2 V_1 = \frac{-\rho}{\varepsilon_0}$$

$$\nabla^2 V_2 = \frac{-\rho}{\varepsilon_0}$$

$$\nabla^2 (V_1 - V_2) = 0$$

$$\nabla^2 V_3 = 0$$

⊛ Note that properties of Laplace (no maxima/minima) apply on V_3 only & not on V_1 and V_2.

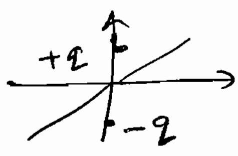
Now V_3 satisfies Laplace equation & has value zero on all boundaries, so $V_3 = 0 \Rightarrow V_1 = V_2$

Corollary: Potential in a volume τ is uniquely determined if (a) charge density throughout the region is specified (b) value of V on all boundary is specified.

$\therefore$ 1st uniqueness theorem valid for Poisson eqn too

$\therefore$ from corollary of 1st Uniqueness theorem, there is only 1 function that meets these requirements. If by a trick, we can discover such a function, then its got to be the right answer.

$\boxed{\text{Trick}}$ Study a completely different problem



It follows that

(1) $V=0$ when $z=0$

(2) $V=0$ when $r\to\infty$

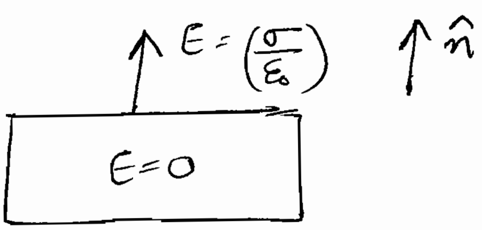
which are the same conditions as previous problem.

$$\therefore\ V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

in the upper region $z \geq 0$

[Note that in lower region $V_1 \neq V_2$ but we don't care]

Induced Surface Charge



$$\sigma = E\epsilon_0$$

$$= -\epsilon_0\left(\frac{\partial V}{\partial n}\right) = -\epsilon_0\left(\frac{\partial V}{\partial z}\right)\Big|_{z=0}$$

Conductor $\vec{E} = \left(\frac{\sigma}{\epsilon_0}\right)\hat{n}$

$$\Rightarrow \sigma(x,y) = \frac{-qd}{2\pi\,[x^2+y^2+d^2]^{3/2}} = \boxed{\frac{-qd}{2\pi\,(x^2+y^2+d^2)^{3/2}}}$$

$\Rightarrow$ Total Induced Charge $Q = \int \sigma\, da$

$$= \int_0^{2\pi}\int_0^{\infty} \sigma\, r\,dr\,d\phi = \boxed{-q}$$

(important to do by Polar Coordinates)

Force & Energy

Since the potential in the vicinity of q is the same as that of analogous problem, so the the field and

$\therefore$ the force is also same

$$\Rightarrow F = -\frac{1}{4\pi\epsilon_0}\cdot\frac{q^2}{(2d)^2}\,\hat{z}$$

Note the plane is grounded. So no free charge apart from $-q$ that is holded by charge q, the remaining $+q$ charge goes to ground. There field below is zero, as $\sigma_{below} = 0$

⊙ Free charge in a grounded conductor flows to the Earth.

$\Rightarrow$ Only half of the energy is there in this problem as compared to analogous problem.

$$W_{actual} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d} = \frac{1}{2} W_{analogous}$$

$$W_{analogous} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2d}$$

W_{actual} can be calculated as

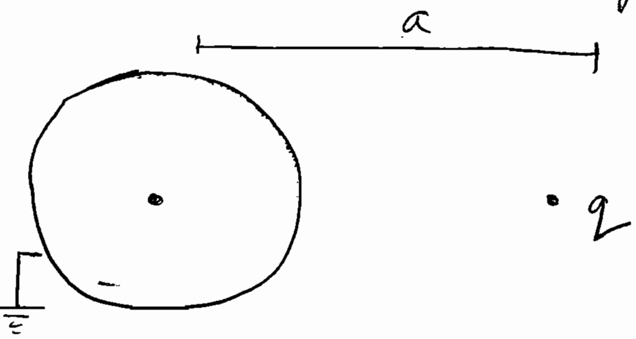
$$W = \int_{+\infty}^{d} F_{ext.} \cdot dx$$

$$= +\int_{\infty}^{d} \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2z)^2}\, dz$$

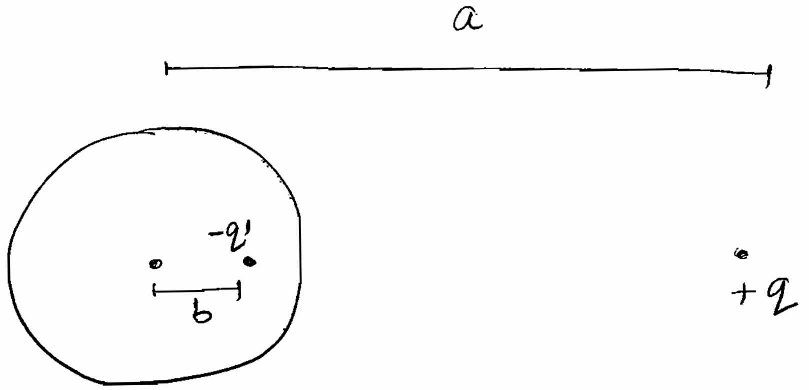
$$= \frac{1}{4\pi\varepsilon}\cdot\frac{q^2}{4}\cdot\left[\frac{1}{\infty} - \frac{1}{d}\right] = \frac{-1}{4\pi\varepsilon_0}\left(\frac{q^2}{4d}\right)$$

* Note that this method is not limited to a single point charge; any stationary charge dist^n near a grounded conducting plane can be treating in the same way by placing opposite charges at mirror image s.t $V=0$ at the plane.

② Aim is to find out Potential outside the sphere.



Analogous Problem



$$-q' = -\left(\frac{R}{a}\right) q$$

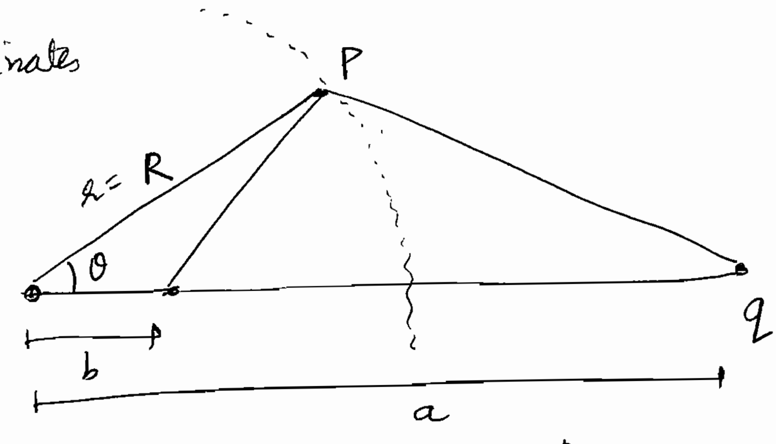
$$b = \left(\frac{R^2}{a}\right)$$

- "a" in denominator in both the cases.

दोनों एक दूसरे से दूर जाएँगे "a" के बढ़ने पर !!

Note that $b < R$. So the image charge is safely inside the sphere; we cannot put image charges in the region where we are calculating the V; that would change ρ and we will solve Poisson equation with wrong ρ.

Use Polar Coordinates.



$$V_P = \frac{1}{4\pi\varepsilon}\left[\frac{q}{\sqrt{R^2+a^2-2Ra\cos\theta}} - \frac{qR}{a\sqrt{b^2+R^2-2bR\cos\theta}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{R^2+a^2-2Ra\cos\theta}} - \frac{q}{\sqrt{\frac{a^2}{R^2}\cdot\frac{R^4}{a^2}+R^2\frac{a^2}{R^2}-2\cdot\frac{R^2}{a}\cdot R\frac{a^2}{R^2}\cos\theta}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{R^2+a^2-2aR\cos\theta}} - \frac{q}{\sqrt{R^2+a^2-2aR\cos\theta}}\right]$$

$$= 0$$

Use similar procedure to calculate $V(r)$ by using r in place of R

$$\vec{\nabla}\cdot\vec{D} = \rho$$

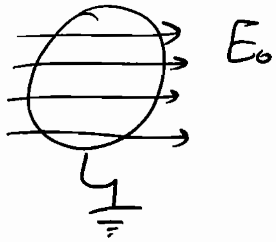
$$\vec{D} = \epsilon\vec{E}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon}$$

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

25) Coulumb → Gauss → ∇·E form → Laplace

$$V(r,\theta) = -E_0 r\cos\theta * \left(1-\frac{a^3}{R^3}\right)$$



32) $\vec{P} = n\vec{p}$

2nd standard problem is : Conducting Sphere and Point Charge

2011 Question

$$V(r,\theta) = -E_0 r\cos\theta\left(1-\frac{a^3}{R^3}\right)$$

$$\vec{E} = -\vec{\nabla}V$$

$$E_r = -\left(\frac{\partial V}{\partial r}\right)$$

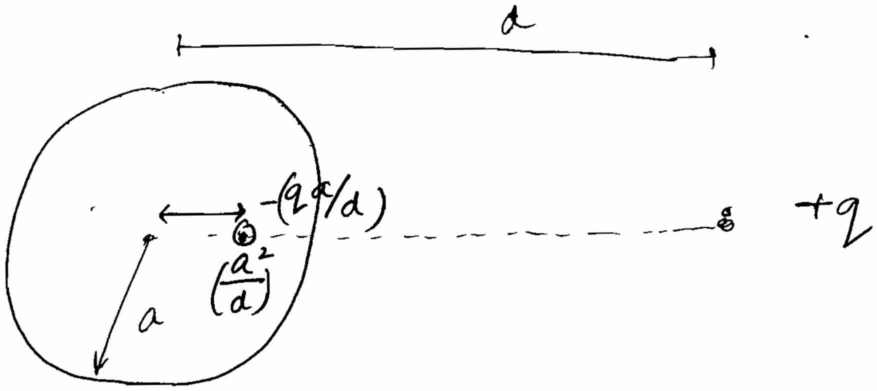
We know, $E = \frac{\sigma}{\epsilon_0}\hat{r}$ $\qquad -\left(\frac{\partial V}{\partial r}\right) = \frac{\sigma}{\epsilon_0}$

← Do not go into detail of σ_{sphere}! IAS बनना है, Physicist नहीं!

$$\sigma = -\frac{q}{4\pi R}(a^2-R^2)(R^2+a^2-2aR\cos\theta)^{-3/2}$$

$$q_{induced} = -\frac{qR}{a} = q'$$

(2) Sphere



$$V(r=a)=0$$

Replace the sphere by another point charge $-q'$ equal to $-\left(\frac{qa}{d}\right)$ placed at $\left(\frac{a^2}{d}\right)$ from centre.

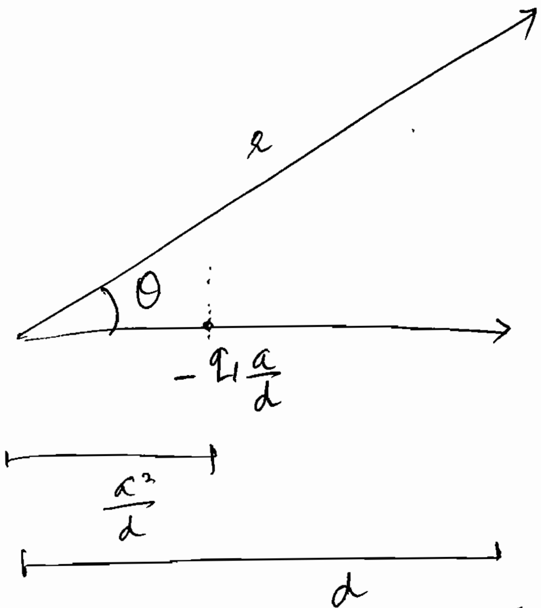
$$\left[\frac{q}{d-a}+\frac{q'}{a-b}\right]=0 \qquad b=\frac{a^2}{d}$$

$$q'=-\frac{q_1 a}{d}$$

$$\Rightarrow \frac{\cancel{q}}{d-a}=\frac{\cancel{q}\,a}{d(a-b)}$$

$$\Rightarrow \cancel{da}-db=\cancel{da}-a^2$$

$$\Rightarrow \boxed{b=\frac{a^2}{d}}$$



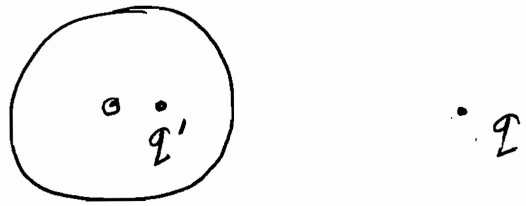
$$V(r,\theta)=\frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+d^2-2rd\cos\theta}}-\frac{(qa/d)}{\sqrt{r^2+(a^2/d)^2-2r\frac{a^2}{d}\cos\theta}}\right]$$

$$\vec{E} = -\vec{\nabla} V$$

$$E_r = -\left(\frac{\partial V}{\partial r}\right)_{r=a} = \left(\frac{\sigma}{\epsilon_0}\right)$$

$$Q = \int_S \sigma \, dA$$

- If not grounded then how to solve by method of images?



Place a charge $= -q'$ at centre of sphere.

Net charge inside conducting sphere $= 0$

Note that in previous question, earthing की वजह से Net charge inside $= 0$

E&M (8)

21/01/2012

- Using Gauss law, we can calculate

$V(r<R)$ inside sphere of charge q

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{R}\left(\frac{3}{2}-\frac{r^2}{R^2}\right)$$

- Using this we can find out, Self Energy

$$W = \int dw = \int V\,dq = \int V \rho\, d\tau$$

$$= \int \frac{1}{4\pi\epsilon_0}\frac{q}{R}\cdot\frac{q}{\frac{4}{3}\pi R^3}\cdot 4\pi r^2\, dr\left(\frac{3}{2}-\frac{r^2}{R^2}\right)$$

$$= \int \frac{1}{4\pi\epsilon_0}\frac{q}{R}\frac{3q}{R}\left(\frac{3}{2}-\frac{r^2}{R^2}\right)$$

$$= \frac{3q^2}{4\pi\epsilon_0 R^2}\left[\frac{3}{2}R-\frac{R^3}{R^2}\right]$$

$$= \frac{3q^2}{4\pi\epsilon_0 R^2}\cdot\frac{1}{2}R = \left(\frac{3Q^2}{8\pi\epsilon_0 R}\right)$$

$$= \int \frac{1}{4\pi\epsilon_0}\frac{3q^2}{R^4}\cdot\left[\frac{3}{2}r^2-\frac{r^4}{R^2}\right]dr$$

$$= \frac{3q^2}{\pi\epsilon_0 R^4}\cdot\left[\frac{3}{2}\frac{R^3}{3}-\frac{R^5}{5R^2}\right]$$

$$= \frac{3q^2}{\pi\epsilon_0 R}\left[\frac{1}{2}-\frac{1}{5}\right] = \frac{3}{20}\frac{Q^2}{\pi\epsilon R}$$

$\boxed{\text{2nd Method}}$

$$E = \frac{1}{2}\epsilon_0\int_{\text{space}} E^2\, d\tau$$

$$= \frac{\epsilon_0}{2}\int\frac{Q^2}{(4\pi\epsilon_0)^2}\cdot\frac{r^2}{R^6}\,d\tau + \frac{\epsilon_0}{2}\int\frac{Q^2}{(4\pi\epsilon_0)^2}\cdot\frac{1}{r^4}\,d\tau$$

Now $d\tau = 4\pi r^2\, dr$

$$\Rightarrow \boxed{E = \frac{3}{5}\frac{Q^2}{(4\pi\epsilon_0)R}}$$

$\boxed{\text{Self Energy}}$

$\boxed{\text{1st Method}}$ $= \int V\,dq$

$$= \int_0^R \frac{1}{4\pi\epsilon_0}\cdot\frac{\rho\cdot\frac{4}{3}\pi r^3}{r}\cdot 4\pi r^2\, dr$$

$$= \frac{3}{20}\frac{Q^2}{\pi\epsilon_0 R}$$

$$= \boxed{\frac{3}{5}\cdot\left(\frac{1}{4\pi\epsilon_0}\right)\frac{Q^2}{R}}$$

Note similarity between it and $\frac{3}{5}G\frac{M^2}{R}$

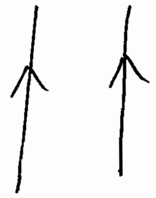
X— Electrostatics —X

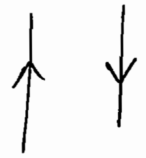
MAGNETOSTATICS

Lorentz Force

$$\vec{F}_{mag} = q(\vec{v} \times \vec{B})$$ Lorentz Force law

- if $v = 0$, no $\vec{F}_{mag}$ $\Rightarrow$ stationary charges experience 0 force

-  Attractive Force if current in same direction

-  Repulsive Force if current in opposite direction

In the presence of both electric & magnetic fields,

$$F = q(\vec{v} \times \vec{B}) + q\vec{E}$$

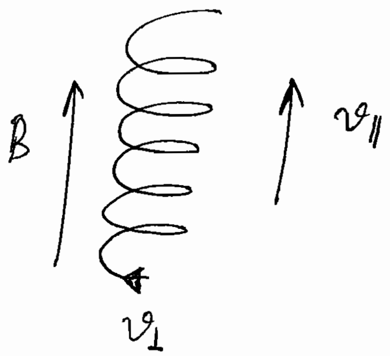
Note that these are empirical laws.

Typical motion under magnetic field is circular with radius as $\left(\frac{mv_\perp}{qB}\right)$

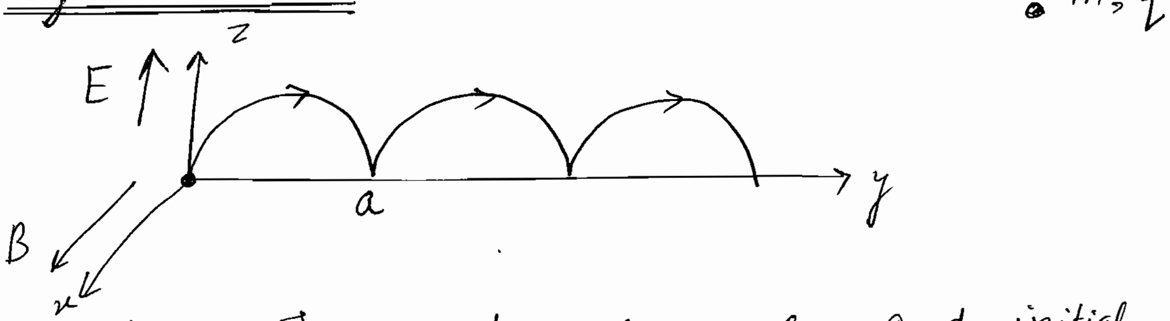
This is the motion in <u>cyclotron</u> - the first of the modern particle accelarators.

Its also a <u>measure of momentum</u> of a particle by measurement of radius.

If it also has $v_\parallel$ component, then helical motion



Cycloid Motion



If $\vec{E}$ and $\vec{B}$ are at right angles and initial particle is at rest at origin. The path followed will be a cycloid.

Initially magnetic force = 0 and electric field accelarates the charge in z-direction. As it picks up speed, the magnetic force pull the charge to right. The faster it goes, stronger the magnetic force becomes, eventually it curves the particle back towards y-axis. At this point, charge moves against ~~y~~ electric force and therefore decelerates, magnetic force decreases, and electric force takes over, bringing the charge to rest at point a. Now the entire process repeats.

Equation of motions comes out :

$$\boxed{(y - R\omega t)^2 + (z - R)^2 = R^2}$$

$$\boxed{R = \frac{E}{\omega B}}$$

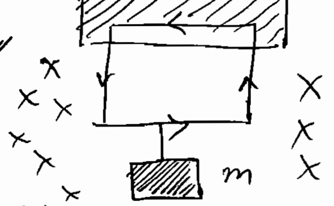
$$\boxed{\omega = \frac{QB}{m}}$$

Note that overall motion is $\perp$ to $\vec{E}$ and $\vec{B}$.

Magnetic Forces do no work.

Even when Magnetic cranes lifts the junk car, still magnetic field does zero work. In every case, when work is perceived to have been done, there is some other agency that does the work.

For current, magnetic force $F_{mag} = \int I (d\vec{l} \times \vec{B})$



- motion of the loop.

Say Magnetic Force > Gravitational Force ⇒ mass m rises up. The work is done by Battery and not magnetic field. Magnetic Force simply redirects the horizontal force of battery into vertical.

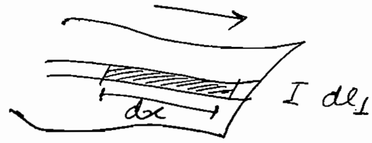
1 Ampere = 1 Coulumb / second

Surface Current Density : $\vec{k}$

$$\boxed{k = \frac{dI}{dl_\perp}}$$: Current per unit width perpendicular to the flow

If (mobile) surface charge density is σ and the velocity is v.

$$dI = \left(\frac{dq}{dt}\right) = \sigma .\ dl_\perp \left(\frac{dx}{dt}\right)$$



$$\Rightarrow \left(\frac{dI}{dl_\perp}\right) = \sigma v$$

$$\Rightarrow \boxed{\vec{k} = \sigma \vec{v}}$$

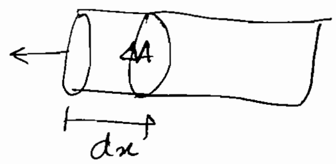
$$\vec{F}_{magnetic} = \int dq\ (\vec{v} \times \vec{B}) = \int \sigma\ da\ (\vec{v} \times \vec{B})$$

$$\boxed{\vec{F}_{mag} = \int (\vec{k} \times \vec{B})\ da}$$

Volume Current Density

$$\boxed{J = \frac{dI}{da_\perp}}$$: Current per unit area perpendicular to the flow.

$$dI = \left(\frac{dq}{dt}\right) = \rho (\Delta A) \frac{dx}{dt}$$



$$\Rightarrow \frac{dI}{dA} = \rho v \quad \Rightarrow \boxed{\vec{J} = \rho \vec{v}}$$

Ohm's law
$\vec{J} = \sigma \vec{E}$

$$\vec{F}_{mag} = \int dq\ (\vec{v} \times \vec{B}) = \int \rho .\ d\tau .(\vec{v} \times \vec{B}) = \int (\vec{J} \times \vec{B})\ d\tau$$

$$\Rightarrow \boxed{\vec{F}_{mag} = \int (\vec{J} \times \vec{B})\ d\tau}$$

Continuity Equation

Now current crossing a surface S can be written as

$$I = \int_S J\, da_\perp = \int_S \vec{J} \cdot d\vec{a}$$

Total charge leaving the volume per unit time is given by

$$\oint_S \vec{J} \cdot d\vec{a} = \int_{\text{Volume}} (\vec{\nabla} \cdot \vec{J})\, d\tau$$

Since the charge is conserved, whatever flow out must come at the expense of that remaining inside.

$$\Rightarrow \int_{\text{volume}} \vec{\nabla} \cdot \vec{J}\, d\tau = -\frac{d}{dt} q_{\text{inside}} = -\frac{d}{dt} \int \rho\, d\tau$$

$$\Rightarrow \int \vec{\nabla} \cdot \vec{J}\, d\tau = \int -\left(\frac{\partial \rho}{\partial t}\right) d\tau$$

$$\Rightarrow \boxed{\nabla \cdot \vec{J} = -\left(\frac{\partial \rho}{\partial t}\right)}$$

Biot-Savart Law

$$B(r) = \frac{\mu_0}{4\pi} \int \frac{I\, d\vec{l} \times \vec{r}}{r^3}$$

for a steady current I.

Note that stationary charges produce electric fields that are constant in time, hence the term electrostatics. Steady currents produce magnetic fields that are constant in time, hence the term magnetostatics.

For most practical applications, magnetostatic apply even for current (households) that varies 50 times a second.

When a steady current flows in a wire, its magnitude remains same all throughout, otherwise charge would be piling up somewhere, and it won't be a steady current. By the same token, $\left(\frac{\partial \rho}{\partial t}\right) = 0$ in magnetostatics, and hence the continuity equation becomes

$$\boxed{\nabla \cdot \vec{J} = 0} \text{ for Magnetostatics.}$$

Note that a moving point charge cannot possibly constitute a steady current

$$\therefore \; B(r) = \frac{\mu_0}{4\pi} q\left(\frac{\vec{v} \times \hat{r}}{r^3}\right) \quad \text{is WRONG.}$$

as Biot Savart's law holds only for steady currents.

Like Electrostatics, Superposition principle applies to Magnetostatics also. If we have a collection of source currents, the net magnetic field is the vector sum of the field due to each of them taken separately.

(*) $I\,d\vec{l} \Leftrightarrow \vec{k} \cdot da \Leftrightarrow \vec{J}\,d\tau \Leftrightarrow q\vec{v}$

Divergence & Curl of B

By some mathematics upon Biot Savart's law, we can get

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$$

Other form of curl of $\vec{B}$, is obtained by using Stokes theorem. This is called AMPERE'S LAW.

$$\int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \mu_0 \int \vec{J} \cdot da = \mu_0 I_{enc} \Rightarrow \boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}}$$

The direction of current taken as ⊕ve should be concomitant with the direction of coverage of loop according to right hand screw rule.

Gauss law in Electrostatics ⇔ Ampere's Law in Magnetostatics

✓ Maxwell Equations for electrostatics : $\vec{\nabla}\times\vec{E} = 0$

(Gauss law) $\vec{\nabla}\cdot\vec{E} = \left(\frac{\rho}{\varepsilon_0}\right)$

Maxwell Equations for magnetostatics : $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ (Ampere's law)

$\vec{\nabla}\cdot\vec{B} = 0$

✓ $\vec{\nabla}\cdot\vec{B} = 0$: It implies that there are no divergence or convergence of magnetic fields; they simply form closed loops or extend out to infinity. There are no point sources for $\vec{B}$ i.e. No Magnetic Monopoles exist.

✓ Typically electric forces are enormously larger than magnetic forces.

Magnetic Vector Potential

Since $\vec{\nabla}\cdot\vec{B} = 0$, we can defines a vector potential $\vec{A}$ such that $\boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$ and $\boxed{\vec{\nabla}\cdot\vec{A} = 0}$

We can state Ampere's law, in terms of $\vec{A}$ as

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{A}(\nabla\cdot\nabla)$$

$$= -\nabla^2\vec{A} = \mu_0\vec{J}$$

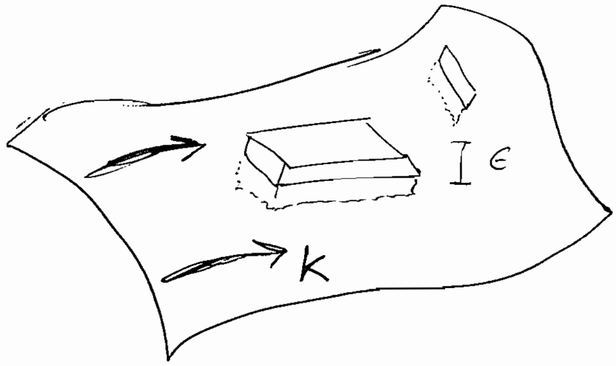
∇^2: scalar का scalar
vector का vector

$$\boxed{\nabla^2\vec{A} = -\mu_0\vec{J}}$$

Ordinarily the direction of $\vec{A}$ is the direction of current

Boundary Conditions for $\vec{B}$

$$\oint \vec{B} \cdot d\vec{a} = 0$$



$$\boxed{B^{\perp}_{above} = B^{\perp}_{below}}$$

$$\oint \vec{B} \cdot d\vec{l} = (B^{\parallel}_{above} - B^{\parallel}_{below})\, l = \mu_0 . I_{enc} = \mu_0 . k l$$

$$\Rightarrow \boxed{B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 k}$$

Thus, the component of $\vec{B}$ that is parallel to the surface but perpendicular to the current is discontinuous in the amount $\mu_0 k$.

$$\Rightarrow \boxed{\vec{B}_{above} - \vec{B}_{below} = \mu_0 (\vec{k} \times \hat{n})}$$

where $\hat{n}$ is a unit vector perpendicular to the surface, pointing upwards.

Just as the electric field is discontinuous at a surface charge, so the magnetic field is discontinuous at a surface current. Only this time, its the tangential component that changes.

MAGNETIC FIELDS IN MATTER

All magnetic phenomenon are due to electric charges in motion. In magnetic materials, on an atomic scale, there are tiny currents, electrons orbiting around nuclei and electrons spinning. For macroscopic purposes, these current loops are so small that we may treat them as magnetic dipoles. Ordinarily they cancel out each other due to random orientations of these dipoles, but when magnetic field is applied, a net alignment of these dipoles occur, and the material becomes magnetically polarized or "magnetized".

Unlike electric polarization, which is almost always in the direction of $\vec{E}$, some materials acquire magnetization parallel to $\vec{B}$ (paramagnetic) and some opposite to $\vec{B}$. (diamagnetic). A few substances (ferromagnetic) retain their magnetization even after external field has been removed. These magnetizations are not determined by the present field but by the whole magnetic "history" of the object.

Torque on a magnetic dipole

$$\vec{\tau} = \vec{m} \times \vec{B}$$

where $\vec{m} = I\vec{a}$

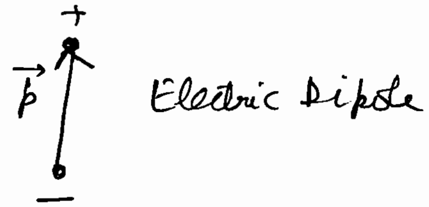
These torque try to align magnetic dipoles parallel to the Magnetic Field and therefore account for paramagnetism. But due to Pauli Exclusion Principle, only unpaired electrons show paramagnetism.

In a uniform field, force on a current loop is zero. but in a non-uniform field, this is no longer the case.

Force on a dipole in Non-Uniform Field

Electric Dipole

Remember that for an electric dipole, we had devised,



$$F = (\vec{p}\cdot\vec{\nabla})\vec{E} \quad — (1)$$

Now consider the argument,

$$U_{dipole} = -\vec{p}\cdot\vec{E}$$

$$\vec{F} = -\vec{\nabla}U = -\vec{\nabla}(-\vec{p}\cdot\vec{E}) = \vec{\nabla}.(\vec{p}\cdot\vec{E}) \quad — (2)$$

Now for electric field,

$$\vec{\nabla}\cdot(\vec{p}\cdot\vec{E}) = \vec{p}\times(\vec{\nabla}\times\vec{E}) + \vec{E}\times(\vec{\nabla}\times\vec{p}) + (\vec{p}\cdot\vec{\nabla})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{p}$$

$$= 0 + 0 + (\vec{p}\cdot\vec{\nabla})\vec{E} + 0$$

$$= (\vec{p}\cdot\vec{\nabla})\vec{E}$$

Hence 1^{st} and 2^{nd} expressions are same.

Also, Mathematically,

$$(\vec{p}\cdot\vec{\nabla})\vec{E} = \sum\left[p_x\left(\frac{\partial E_x}{\partial x}\right) + p_y\left(\frac{\partial E_x}{\partial y}\right) + p_z\left(\frac{\partial E_x}{\partial z}\right)\right]\hat{i}$$

$$\vec{\nabla}\cdot(\vec{p}\cdot\vec{E}) = \sum\left[p_x\left(\frac{\partial E_x}{\partial x}\right) + p_y\left(\frac{\partial E_y}{\partial x}\right) + p_z\left(\frac{\partial E_z}{\partial x}\right)\right]\hat{i}$$

From $\vec{\nabla}\times\vec{E} = 0$, we know

$$\frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z} \qquad \frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x} \qquad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$$

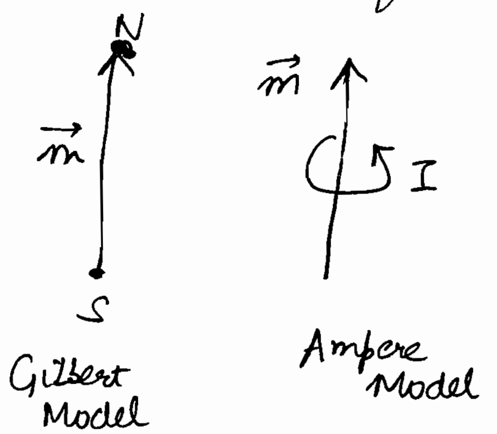
$\therefore$ the two are same.

Preferable way of writing is $(\vec{p}\cdot\vec{\nabla})\vec{E}$

Magnetic Dipole

Since $\vec{\nabla} \times \vec{B} \neq 0$, we cannot write in the preferable form

$$\therefore \quad \boxed{F = \nabla(\vec{m} \cdot \vec{B})}$$



As we can see, the gilbert model, though incorrect ($\because$ there are no magnetic monopoles), gives approximation to the true Ampere Model and can be applied in many situations. Application of Gilbert Model gives the advantage of using results of electric dipole by replacing of

$$\left.\begin{array}{l} \vec{p} \text{ by } \vec{m} \\ (1/\varepsilon_0) \text{ by } \mu_0 \end{array}\right\}$$

H.C. Verma की chapter इसी पर based है !!

Note again that there is no such thing as single magnetic North Pole or South Pole. If you break a bar magnet in half, you don't get a North Pole in one hand and south pole in ~~other~~; you get two complete magnets. Magnetism is not due to magnetic monopoles but due to moving electric charges. Magnetic dipoles are tiny current loops and its an extraordinary thing, really, that the formulas involving $\vec{m}$ bear any resemblance at all to the corresponding formula for $\vec{p}$.

⊛ Magnetization

Whatever be the cause of magnetization, we describe the state of magnetic polarization by vector quantity $\vec{M}$ called Magnetization.

$\vec{M}$: magnetic dipole moment per unit volume.

In general when a sample is placed in a region of non-uniform field, a paramagnet is attracted into the field while the diamagnet is repelled away.

Ampere's law for magnetized materials

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_{total} \quad \text{(Ampere's law)}$$

Now $\vec{J}_{total} = \vec{J}_{free} + \vec{J}_{bound}$

and $\vec{J}_{bound} = \vec{\nabla} \times \vec{M}$

[Note that similar derivation can be done for Vector Magnetic Potential to get

$$\vec{J}_{bound} = \vec{\nabla} \times \vec{M}$$

$$\vec{K}_{bound} = \vec{M} \times \hat{n}$$

[Replace ρ by $\vec{J}$, σ by $\vec{K}$, $(\cdot)$ by $(\times)$ and make all signs ⊕ve, $\vec{P}$ by $\vec{M}$]

$$\Rightarrow \vec{\nabla} \times \left[\frac{\vec{B}}{\mu_0} - \vec{M}\right] = \vec{J}_{free}$$

Define $\dfrac{\vec{B}}{\mu_0} - \vec{M} = \vec{H}$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{H} = \vec{J}_{free}}$$

$$\boxed{\oint \vec{H} \cdot d\vec{l} = I_{fenc}}$$

⊛ Many authors call H as Magnetic Field
B as Flux Density

Boundary Conditions

$$H^{\perp}_{above} - H^{\perp}_{below} = -\left(M^{\perp}_{above} - M^{\perp}_{below}\right) \quad \text{(from def}^n\text{)}$$

$$H^{\parallel}_{above} - H^{\parallel}_{below} = \vec{K}_f \times \hat{n} \quad \text{(from Ampere's law)}$$

Linear Media

$$\vec{M} = \chi_m \vec{H}$$

χ_m : magnetic susceptibility

$\oplus$ve for paramagnets

$\ominus$ve for diamagnets

Now $$\boxed{\vec{B} = \mu_0 (\vec{H} + \vec{M})}$$

$\Rightarrow$ for linear media $\vec{B} = \mu_0 [\vec{H} + \chi_m \vec{H}]$

$$\Rightarrow \boxed{\vec{B} = \mu_0 (1 + \chi_m) \vec{H}}$$

$\mu = \mu_0 (1 + \chi_m)$: Permeability of the material

$\mu = \mu_0 \mu_r$: μ_r : relative permeability

For space $\boxed{\vec{B} = \mu_0 \vec{H}}$

Also for linear [redacted] magnetic media,

$$J_{bound} = \vec{\nabla} \times \vec{M} = \vec{\nabla} \times \chi_m \vec{H} = \chi_m \vec{J}_{free}$$

$$\boxed{J_{bound} = \chi_m \vec{J}_{free}}$$

i.e. <u>unless a free current flows through the material, all bound current will be at the surface.</u>

Magnetic Scalar Potential

The differential form of Ampere's law ($\vec{\nabla}\times\vec{H} = \vec{J}_f$) shows that $\vec{H}$ is not a conservative field and thus cannot be written in terms of a gradient of a scalar potential. But there are certain regions of space called "Simply Connected" (regions outside of the current sources in which there is no possibility of any closed path being linked with a current), where magnetic field $\vec{H}$ could be described as a conservative field and can be expressed as the gradient of a scalar potential function

$$\vec{H} = -\nabla\phi_m \quad \& \quad \vec{\nabla}\times\vec{H} = 0$$

s.t.

We know $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{\nabla}\cdot\mu_0(\vec{H}+\vec{M}) = 0 \Rightarrow -\vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot\vec{M} \Rightarrow \boxed{\nabla^2\phi_m = \vec{\nabla}\cdot\vec{M}}$

From some mathematics, it can be shown that

$$\boxed{\phi_m = \frac{I}{4\pi}\Omega}$$

Ω : solid angle subtended at the point P by the current loop.

Similar to electric dipole, magnetic dipole also has Potential energy

$$U = -\vec{m}\cdot\vec{B}$$

$$\vec{A} = \frac{\mu I}{4\pi} \int \frac{\vec{dl}}{r}$$

Correct formula

MAGNETOSTATICS

Magnetic Property

μ Permeability

μ_r Relative Permeability

$\mu = \mu_r \mu_0$

Non Magnetic Material
$\mu_r = 1$

It gets Magnetized

$$\boxed{\vec{B} = \mu \vec{H}}$$

Applied Magnetic Field
$\vec{H}$

$$\vec{B} = \mu \vec{H}$$
$$= \mu_0 (\vec{H} + \vec{M})$$
$$\Rightarrow (\mu - \mu_0)\vec{H} = \mu_0 M$$
$$\Rightarrow \mu_r = 1 + \frac{\vec{M}}{\vec{H}}$$
$$= 1 + \chi$$
$$\chi_m = \frac{M}{H}$$

Alignment of dipoles

Equivalent Electrical Property

ϵ Permittivity

ϵ_r Relative Permittivity

$\epsilon = \epsilon_r \epsilon_0$

Air $\epsilon_r = 1$

It gets Polarized

$$\boxed{\vec{D} = \epsilon \vec{E}}$$

Applied Electrical Field
$\vec{E}$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$
$$\vec{P} = (\epsilon - \epsilon_0)\vec{E}$$
$$= (k-1)\epsilon_0 \vec{E} \qquad (k = \epsilon_r)$$
$$\chi_e = \frac{\vec{P}}{\epsilon_0 \vec{E}}$$

Alignment of charges

$\vec{H}$ applied, then $\vec{M}$ occurs due to alignment of dipoles, then $\vec{B}$.

$\Rightarrow$ $\vec{M}$ and $\vec{B}$ lag behind $\vec{H}$

$\Rightarrow$ this lagging is called HYSTERESIS.

$$\boxed{\vec{M} = \chi \vec{H}}$$

For ferromagnetic materials,

$\mu >> \mu_0$ i.e. $\chi >> 1$

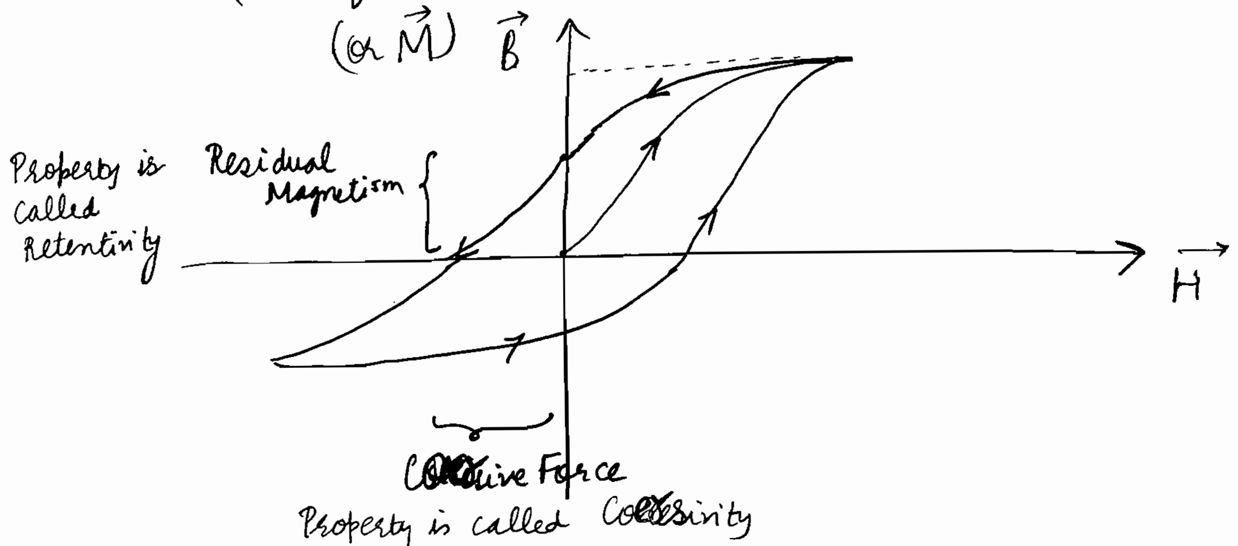
For paramagnetic materials,

$\mu > \mu_0$ i.e. $\chi > 0$

In ferromagnetic Materials, $\vec{M}$ and $\vec{B}$ lag behind $\vec{H}$, which leads to hysteresis !!

When ferromagnetic material is undergoing cycles of magnetization & demagnetization, there is dissipation of energy due to hysteresis.

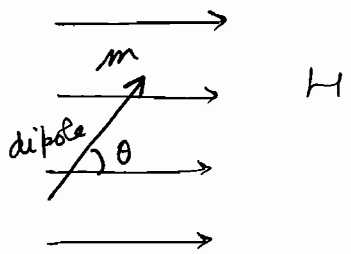
$\boxed{\text{Loss of Energy} \propto \quad H\, dm}$



Electromagnetics [S. Singh]

- $\tau = \vec{m} \times \mu \vec{H}$

$= \mu m H \sin\theta$



- $dW = \tau \, d\theta$

$= \mu m H \sin\theta \, d\theta$

$W = \mu H \sum m \sin\theta \, d\theta$

$= \mu H \sum -m \, d(\cos\theta)$

$W = \mu H \, -d\sum m \cos\theta$

$\Rightarrow \left(\frac{W}{V}\right) = -\mu \vec{H} \cdot d\vec{M}$

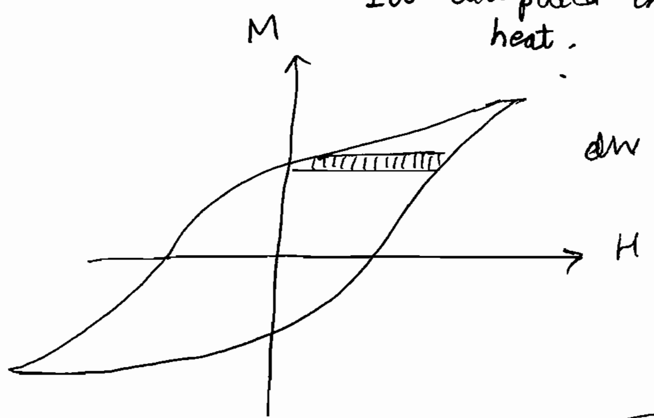
$W = \oint \vec{H} \cdot d\vec{M}$

$=$ Area of loop !!

$\Rightarrow$ $\boxed{\text{Hysteresis loss} = [\text{Area of MH loop}] * \mu_0}$

$$\boxed{\begin{aligned} &\text{Hysteresis loss per unit volume per cycle} \\ &= \text{Area of BH Hysteresis loop} \\ &= \oint H \cdot dB = \oint H \cdot \mu_0 (dH + dM) \\ &= \mu_0 \oint H \, dM \quad (\text{WARBURG'S LAW}) \end{aligned}}$$

Its dissipated in the form of heat.



$dW = -H \, dM$

$W = \oint \vec{H} \cdot \left(\frac{d\vec{B}}{\mu_0} - d\vec{H}\right)$

$= \frac{1}{\mu_0} \oint \vec{H} \cdot d\vec{B}$

$\uparrow \left(\oint \vec{H} \cdot d\vec{H} = 0\right)$

(यह क्यों होता है, दिमाग नहीं)

Soft Iron : low Area of Hysteresis loop → Transformer

Hard Iron : high Area of Hysteresis loop → Permanent Magnet eg. Steel

Also since Work Done = Some constant times HdM equivalence used in Adiabatic Demagnetization was valid

Also in CGS units, $\mu_0 = 1$ and its dimensionless.

Numerical

Mass$_{core}$ = 10 kg

f = 50 Hz

Time = 1 hour

d_{iron} = 1.5×10^3 kg/m^3

Area of Hysteresis loop = 6×10^3 erg/cc

Find out hourly loss of energy.

Assuming loss is only due to hysteresis:

No. of cycles = $50 \times 60 \times 60$

$$\text{Volume} = \frac{10}{1.5 \times 10^3}$$

Loss = Area × Volume × cycles

$$= 6 \times \cancel{10^3} \times \frac{10}{1.5 \times \cancel{10^3}} \times 50 \times 60 \times 60 \times 10^6$$

$$= 4 \times 5 \times 36 \times 10^{10} \text{ ergs}$$

$$= \underline{7.2 \times 10^{12} \text{ ergs}}$$

Current Electricity

Preliminaries : Kirchoff's laws : $\Sigma I = 0$

$\Sigma V = 0$

Biot's Savert's law

Ampere's law

Faraday laws of EM Induction < Self Induction, Mutual Induction

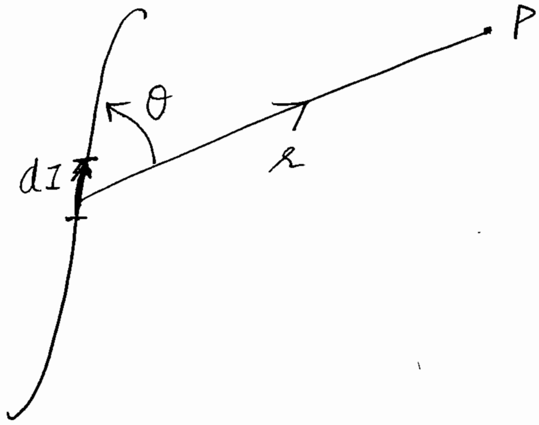
Lenz's law

Transient Circuits : LC, RC, LCR

AC Circuits

Biot Savert's law

- I is scalar as it is not added by vector law of addition. But dI (current element) is considered as vector quantity.



Current Element has ability to attract.
Larger the distance, lesser the attractive power.

It is noted

$$dB \propto I\, dl \sin\theta$$

$$dB \propto \frac{1}{r^2}$$

$$dB = \frac{\mu_0}{4\pi} \frac{I\, dl}{r^2} \sin\theta = \frac{\mu_0}{4\pi} \frac{I\, dl\, r \sin\theta}{r^3}$$

$$\boxed{\vec{dB} = \frac{\mu_0}{4\pi} I \left[\frac{\vec{dl} \times \vec{r}}{r^3}\right]}$$

<empirical law>
Biot Savertz's law

direction of $\vec{B}$ given by Right Hand Rule for cross products

Let us write $\boxed{\vec{H} = -\vec{\nabla}\phi_m}$

Magnetostatic Scalar Potential

[I does not vary wrt time]

We can also write

$$\boxed{\vec{B} = \nabla \times \vec{A}}$$

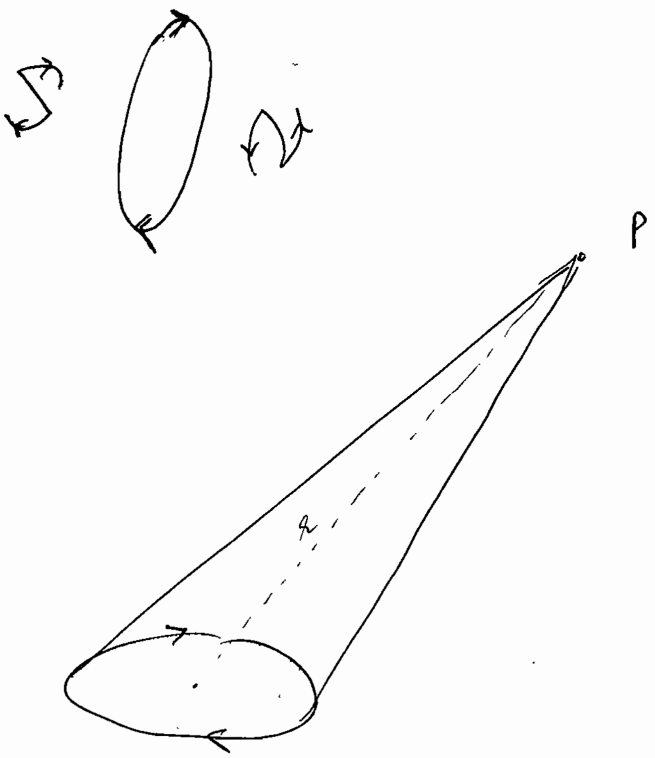
Magnetostatic Vector Potential

⊙ Remember that ϕ (Scalar Potential) and $\vec{A}$ (Vector Potential) are just Mathematical Constructs

Take $\boxed{\phi_m = \frac{I}{4\pi}\,\Omega}$

Solid Angle subtended by current carrying loop.

✓ Note the every current carrying loop is a dipole.



Here $\vec{A}$ is the area vector

$$\boxed{\Omega = \frac{\vec{A}\cdot\vec{r}}{r^3}} \quad ; \quad \boxed{\phi_m = \frac{I}{4\pi}\,\frac{\vec{A}\cdot\vec{r}}{r^3}}$$

$$\boxed{\vec{H} = -\vec{\nabla}\left\{\frac{I}{4\pi}\cdot\frac{\vec{A}\cdot\vec{r}}{r^3}\right\}}$$

We know $d\Omega = (\nabla\Omega)\cdot dl$

$$\vec{H} = -\frac{\mu_0 I}{4\pi} \int \frac{dl \times r}{r^3} = \frac{-\mu_0 I}{4\pi} \int dl \times \vec{\nabla}\left(\frac{1}{r}\right)$$

$$\phi_m = \frac{\mu_0 I}{4\pi} \frac{\vec{A} \cdot \vec{r}}{r^3}$$

$$\phi_m = \frac{\mu_0 (I\vec{A}) \cdot \vec{r}}{4\pi \; r^3} = \left(\frac{\mu_0}{4\pi}\right) \frac{\vec{m} \cdot \vec{r}}{r^3}$$

Note that $\boxed{\phi \Leftrightarrow V_{dipole}}$ ✓ (Just μ_0 is missing : rest is same)

Hence all the results derived for dipole electrical are valid for ϕ_m, corresponding to that of V.

$$\vec{H} = -\vec{\nabla}\phi_m$$

(2 scalars को अलग अलग करो)

$$= -\frac{\mu_0}{4\pi} \vec{\nabla}\left(\frac{\vec{m} \cdot \vec{r}}{r^3}\right)$$

$$= -\frac{\mu_0}{4\pi}\left[\nabla \cdot (\vec{m} \cdot \vec{r}) \frac{1}{r^3} + (\vec{m} \cdot \vec{r}) \cdot \nabla\left(\frac{1}{r^3}\right)\right]$$

$$\Rightarrow \boxed{\vec{H} = +\frac{\mu_0}{4\pi}\left[\frac{3(\vec{m} \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right]} \checkmark$$

$U = -\vec{m} \cdot \vec{B}$ & $\vec{B} = \mu_0 \vec{H}$ (for vacuum)

$$\Rightarrow \boxed{U = \frac{\mu_0}{4\pi}\left[\frac{\vec{m}_1 \cdot \vec{m}_2}{r^3} - \frac{3(\vec{m}_1 \cdot \hat{r})(\vec{m}_2 \cdot \hat{r})}{r^3}\right]}$$

~~$\vec{B} = -\vec{\nabla}\phi_m$~~

~~We know $\vec{\nabla}\cdot\vec{B} = 0$~~

~~$\Rightarrow \boxed{\nabla^2 \phi_m = 0}$~~

i.e. Monopoles do not exist

experimentally proven

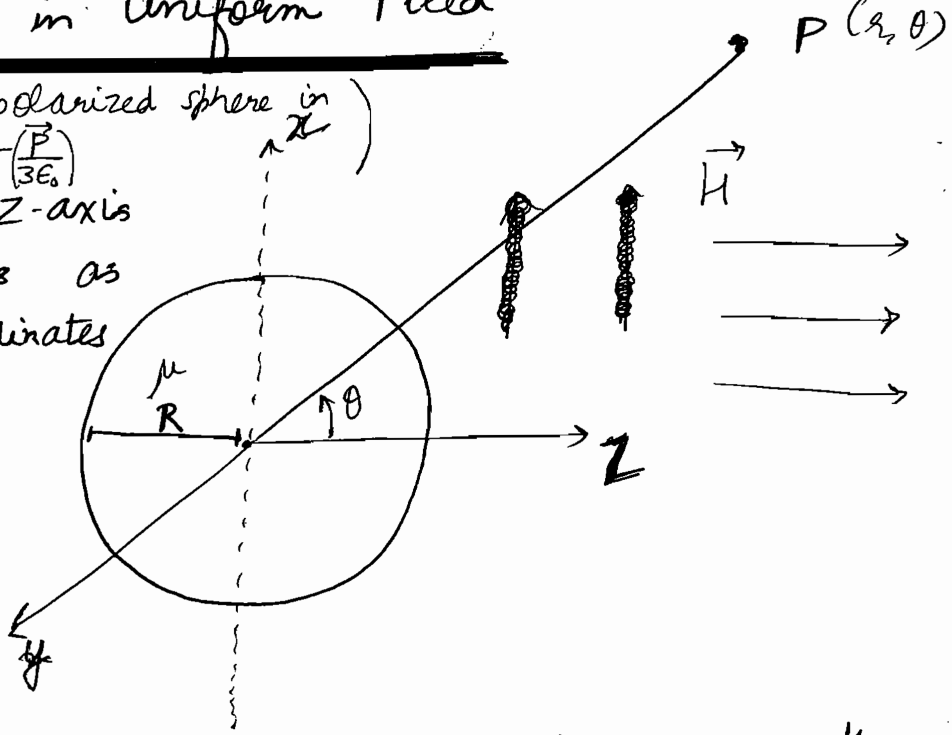
$$\oint_s \vec{B}\cdot\vec{ds} = 0$$

$$\Rightarrow \int_v \vec{\nabla}\cdot\vec{B}\, dv = 0 \quad \Rightarrow \quad \boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

Magnetized Sphere in Uniform Field

(direct equivalence to polarized sphere in electric field $\vec{E}_p = -\left(\frac{\vec{P}}{3\epsilon_0}\right)$)

→ take $\vec{H}$ along z-axis and θ from z-axis as in spherical coordinates



$\vec{H}$ is uniform

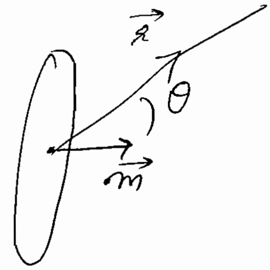
$\vec{M} = I\vec{A}$

$\vec{B} = \mu_0 \vec{H}$ (at P, the free space)

$\vec{B} = \mu H$ (inside the sphere)

① there is current on the surface of the sphere

② current carrying loop is behaving as Magnetic dipole



$r > R$: $\phi_m = \frac{\cancel{\mu_0}}{4\pi}\frac{\vec{m}\cdot\vec{r}}{r^3} = \frac{\cancel{\mu_0}\, m\cos\theta}{4\pi r^2}$

We know $M = \frac{m}{\left(\frac{4}{3}\pi R^3\right)}$

Part A No external field

$\Rightarrow \phi_m = \frac{\cancel{\mu_0}}{4\pi}\cdot\frac{m\cos\theta}{r^2} = \frac{\cancel{\mu_0}}{\cancel{4\pi}}\cdot\frac{\cancel{4\pi} M R^3}{3}\frac{\cos\theta}{r^2}$

$$\boxed{\phi_m = \frac{\cancel{\mu_0} M R^3}{3}\left(\frac{\cos\theta}{r^2}\right)}$$

(∵ ~~out side there~~ is no free currents

OUTSIDE

$\vec{H}_{outside} = -\vec{\nabla}\phi_m$

$\vec{B}_{out\,side} = \mu_0 \vec{H}$

$\vec{B}(r>R) = -\vec{\nabla}\phi_m$

$B_r = -\frac{d\phi_m}{dr}$ $\quad B_\theta = -\frac{1}{r}\left(\frac{d\phi_m}{d\theta}\right)$

$\vec{B} = (B_r\hat{r} + B_\theta\hat{\theta})$

$\vec{H}_\theta = \left(\frac{B_\theta}{\mu_0}\right)\hat{\theta}$ $\quad \vec{H} = \vec{H}_\theta + \vec{H}_r = \left(\frac{B_r}{\mu_0}\hat{r} + \frac{B_\theta}{\mu_0}\hat{\theta}\right)$

$\vec{H}_r = \left(\frac{B_r}{\mu_0}\right)\hat{r}$

$\vec{H}_{out} = -\vec{\nabla}\left(\frac{MR^3\cos\theta}{3r^2}\right)$

$= -\frac{MR^3}{3}\vec{\nabla}\left(\frac{\cos\theta}{r^2}\right)$

$= -\frac{MR^3}{3}\left[\frac{-2\cos\theta\,\hat{r}}{r^3} - \frac{\sin\theta\,\hat{\theta}}{r^3}\right]$

$\vec{H}_{outside} = \frac{MR^3}{3r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$

$r < R$: We know $\nabla^2\phi_m = 0$

& $\phi_{in} = \phi_0\big|_{r=R} = \frac{\mu_0 M}{3} R\cos\theta$

let $\phi_{in} = \frac{\cancel{\mu_0} M}{3} z$

$$\vec{B} = -\vec{\nabla}\phi_{in}$$

$$= -\frac{d\phi_m}{dz}$$

$$= -\left(\frac{\mu_0 M}{3}\right)$$

$$\vec{H}_{Total} = \frac{B}{\mu} = -\left(\frac{\vec{M}}{3}\right)$$

$$\boxed{\vec{H} = \vec{H}_0 - \frac{\vec{M}}{3}}$$

$$\phi_m(r=R) = \frac{M}{3} R\cos\theta$$

INSIDE

$$\phi_m(\text{inside}) = \frac{Mz}{3}$$

$$\Rightarrow \vec{H}_{inside} = -\vec{\nabla}\phi_m$$

$$= -\frac{\vec{M}}{3}\hat{z}$$

This method is justified by the Uniqueness theorem

$$\vec{B}_{in} = \mu_0(\vec{H}+\vec{M})$$

$$= \frac{2}{3}\mu_0\vec{M}$$

Now let us also apply an external field H_0

$$H_{total} = H_{applied} + H_{magnetization}$$

Part B

External Field H_0

$$\Rightarrow \boxed{\vec{H}_{total\,inside} = \vec{H}_0 - \frac{\vec{M}}{3}}$$

if linear medium, we have

$$\vec{M} = (\mu_r - 1)H_{Total}$$

$$= (\mu_r - 1)\cdot\left(H_0 - \frac{\vec{M}}{3}\right)$$

$$\vec{D} = \epsilon_0\vec{E}_T + \vec{P}$$

$$\vec{B} = \mu_0(\vec{H}_{total} + \vec{M})$$

$$\Rightarrow \vec{M}\left[1 + \frac{\mu_r - 1}{3}\right] = (\mu_r - 1)H_0$$

$$\Rightarrow \vec{M} = \frac{3(\mu_r - 1)H_0}{(\mu_r + 2)}$$

$$\Rightarrow \boxed{\vec{H}_{total} = \vec{H}_0 - \frac{(\mu_r - 1)}{(\mu_r + 2)}\vec{H}_0 = \frac{3\vec{H}_0}{(\mu_r + 2)}}$$

$$B = \mu_0\cdot(\vec{H}_{t0} + \vec{M})$$

Electrodynamics

Ohm's law

1st form $\boxed{\vec{J} = \sigma \vec{E}}$ σ : Conductivity of the medium (No Proof)

2nd form $\boxed{V = IR}$ R : resistance of the medium (No Proof shown from cylindrical example)

Now we know $\vec{\nabla}\cdot\vec{J} = -\left(\frac{\partial \rho}{\partial t}\right)$ continuity equation

$\vec{\nabla}\cdot\vec{E} = \left(\frac{\rho}{\epsilon_0}\right)$ gauss law

⇒ For steady currents and uniform conductivity,

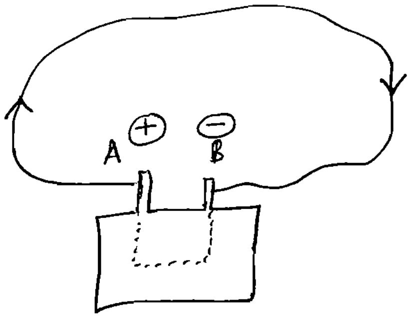
$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\sigma}\vec{\nabla}\cdot\vec{J} = 0$$ (since $\vec{\nabla}\cdot\vec{J} = 0$ for steady currents)

Now, from gauss law, $\rho = 0$

Therefore the charge density is zero ; any unbalanced charge resides on the surface. Therefore, ~~this~~ is true not only for stationary charges but also for steady currents.

⇒ laplace equation holds good within homogenous ohmic material.

Electromotive force ($\mathcal{E}$)



$$\mathcal{E} = V_A - V_B = -\int_b^a \vec{E}\cdot\vec{dl}$$

$$= \int_a^b \vec{E}\cdot\vec{dl}$$

$$= \int_b^a \vec{f_s}\cdot\vec{dl}$$

f_s : external force per unit charge provided by battery.

There are two forces that are involved in driving current around the loop :

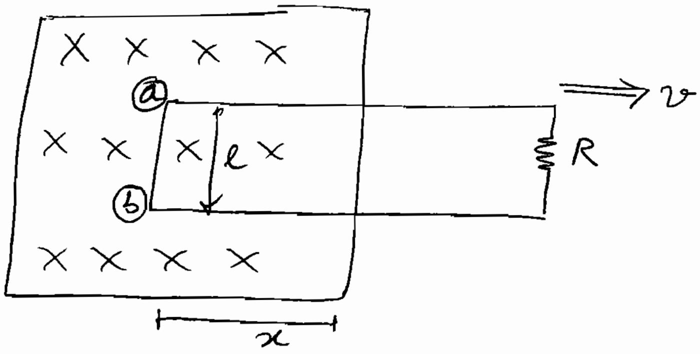
① the source f_s, which is ordinarily

confined to one portion of the loop (eg. battery)

(2) Electrostatic force, which serves to smooth out the flow and the communicate the influence of the source to distant parts of the circuit.

Motional EMF

(1) loop is moving towards right with velocity v



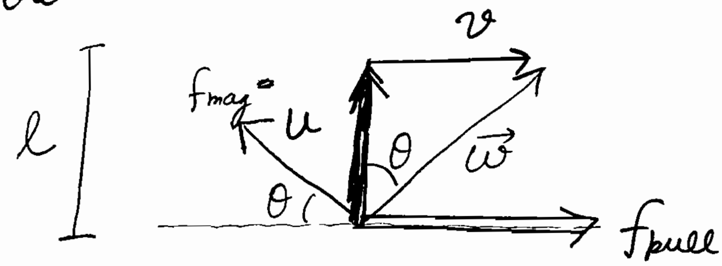
$$f_{mag} = \frac{q.vB}{q} = vB$$

$$\mathcal{E} = \int_b^a f_{mag}.\, dl = \underline{\underline{vBl}}$$

this is called Motional EMF.

Work Done in Motional EMF

Although magnetic force is responsible for establishing the emf, it certainly does not work: magnetic forces never do work. The person who pulls the loop does the work.



u: vertical component of velocity of charges

$$f_{pull} = f_{mag}\cos\theta = uB$$

Work done per unit charge,

$$\int f_{pull}.\, dl = uB.\frac{l}{\cos\theta}\sin\theta = \underline{\underline{Bl.v}} = \mathcal{E}$$

Note that f_{pull} contributes nothing to EMF $\because$ it is $\perp$ to wire while f_{mag} contributes nothing to work $\because$

its perpendicular to motion of charge. This is because, to calculate EMF, we integrate around the loop at one instant, but to calculate work done, we need to follow a charge in its path of motion.

The motional EMF in the general case is written as

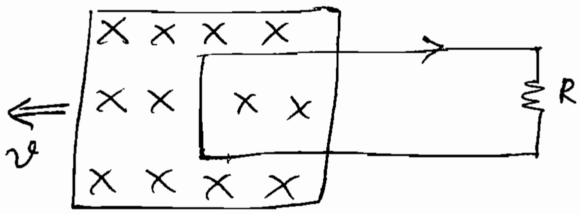
$$\boxed{\mathcal{E} = -\left(\frac{d\phi}{dt}\right)}$$

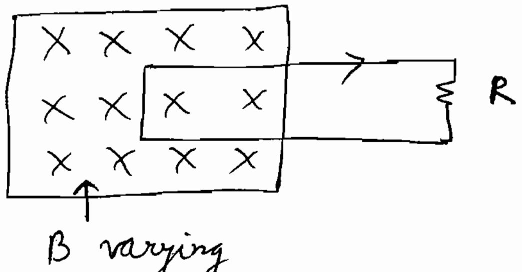
<Proof on P-315 D.Griffiths>

where ϕ is the flux cutting through the loop.

Electromagnetic Induction

Now consider the other two experiments of Faraday (apart from Motional EMF)





We have induced EMF in both these cases, even though, no charge is moving. So what applies for ??

In Both these field is changing (In '1', at a fixed location, field is changing with time as the magnet passes by)

Faraday suggested $\boxed{\text{A changing magnetic field induces an electric field}}$

$$\mathcal{E} = \oint \vec{E}\cdot d\vec{l} = -\left(\frac{d\phi}{dt}\right)$$

[Motional emf formula is always valid]

**
$$\mathcal{E} = -\left(\frac{d\phi}{dt}\right)$$

$$\Rightarrow \oint \vec{E}\cdot d\vec{l} = -\int\left(\frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{a}$$

$$\boxed{\vec{\nabla}\times\vec{E} = -\left(\frac{\partial \vec{B}}{\partial t}\right)}$$

ज़्यादा दिमाग लगाने की ज़रूरत नहीं है !!

Direction of current flow is given by Lenz's law :

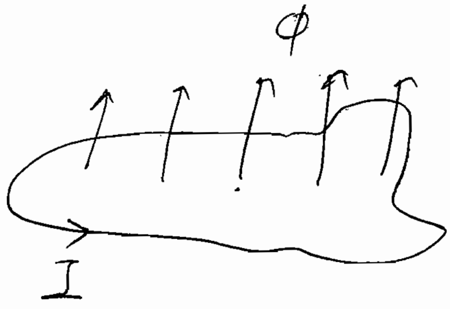
$\boxed{\text{Nature abhors a change in flux}}$

Inductance

Flux is proportional to the current that produces the flux

For a single loop,

current I will produce some flux in the loop



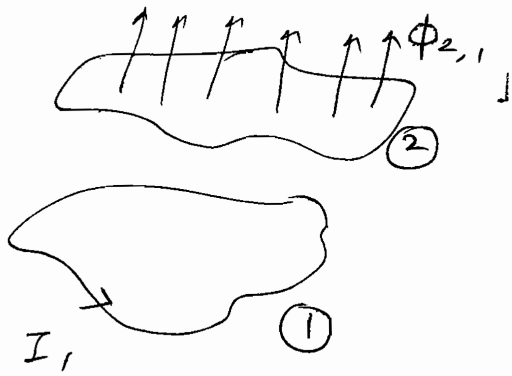
$\phi \propto I$

$$\boxed{\phi = L I}$$

L : self inductance (depends upon geometry of the loop)

For a double loop,

due to current flowing in loop 1, a flux will pierce through loop 2



$$\boxed{\phi_{2,1} = M_{2,1} \, I_1}$$

M_{21} : Mutual inductance of two loop

depend on geometry & relative position of two loops

$\boxed{M_{21} = M_{12}}$

Whatever the shapes and positions of the loops, the flux through 2 when we run a current I through 1 is identical to flux through 1 when we send the same current I around 2.

Now when the currents are varied, EMF is induced in both the situations,

$$\mathcal{E}_2 = -\frac{d\phi_2}{dt} = -M\left(\frac{dI_1}{dt}\right)$$

$$\mathcal{E} = -L\left(\frac{dI}{dt}\right)$$

Inductance is measured in Henries (H). 1 H = 1 V-sec/Amp

* Note that in inductance, flux is always taken as $N\phi$ where N are no. of loops and ϕ is flux through single loop. (flux linkage)

* ⊖ve sign in induced EMF means <u>Back EMF</u>. Its acts opposite to the EMF that tries to alter the current in the wire.

Energy in Magnetic Fields

$$W = \frac{1}{2} L I^2$$ W: Work done to build up current I.

Also, $$W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\, d\tau$$ This is the energy stored in magnetic field.

W: Work done to set up magnetic field B.

* It feels strange that though magnetic fields themselves do no work, energy is required to set up magnetic fields.

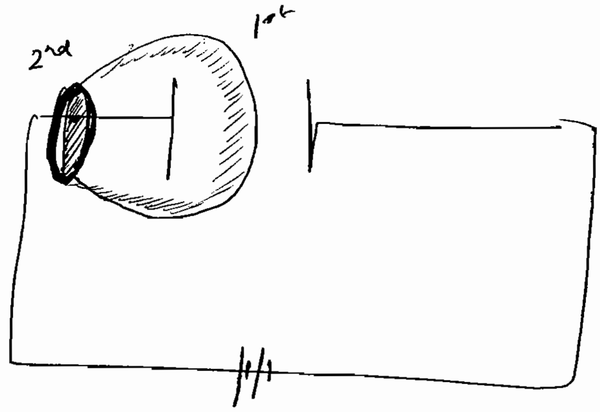
Point is that producing a magnetic field, where previously there was none, requires changing the field, and a changing magnetic field induces Electric field which of course does work.

In the beginning there is no $\vec{E}$ and at the end there is no $\vec{E}$; but in between while $\vec{B}$ is building up, there is an $\vec{E}$ and against this that the work is done.

Modification in Ampere's law

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

Consider a charging capacitor ckt. Let's construct an Amperian loop



From 1st surface,

$$\oint \vec{B}\cdot d\vec{l} = 0$$

From 2nd surface

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

This is ~~a~~ contradiction.

Also from Ampere's law,

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

$$\Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0 \quad \text{(divergence of curl = 0)}$$

$$\Rightarrow \mu_0 (\vec{\nabla}\cdot\vec{J}) = 0$$

Now $\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\varepsilon_0 \vec{\nabla}\cdot\vec{E}) = -\vec{\nabla}\cdot\left(\varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$

Therefore the contradiction is removed by introducing another electric field, s.t.

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \left(\frac{\partial \vec{E}}{\partial t}\right)$$

Therefore,

A changing electric field induces a magnetic field

This theoretical argument was confirmed by HERTZ EXPERIMENTS.

Now this new current density was called Displacement Current Density,

$$\boxed{\vec{J}_d = \varepsilon_0 \left(\frac{\partial \vec{E}}{\partial t}\right)}$$

Now $\oint \vec{B}\cdot d\vec{l} = \int \mu_0 \cdot \varepsilon_0 \cdot \left(\frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{a}$

$= \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \cdot \int \vec{E}\cdot d\vec{a}$

$= \mu_0 \frac{\partial}{\partial t} q_{enc} = \mu_0 I_d$

Maxwell Equations

General form

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\left(\frac{\partial \vec{B}}{\partial t}\right)$$

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \left(\frac{\partial \vec{E}}{\partial t}\right)$$

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

Maxwell's equations tell us how charges and currents produce fields.

Reciprocally, the force law tell us how fields affect charges

Maxwell Equations in Matter (particular case of linear materials)

✓ $\vec{\nabla}\cdot\vec{D} = \rho_f$

✓ $\vec{\nabla}\cdot\vec{B} = 0$

✓ $\vec{\nabla}\times\vec{E} = -\left(\frac{\partial \vec{B}}{\partial t}\right)$

✓ $\vec{\nabla}\times\vec{H} = \vec{J}_f + \left(\frac{\partial \vec{D}}{\partial t}\right)$

We write them because inside polarized or magnetized matter there will be accumulations of "bound" charge and current over which we have no direct control. Its nice to formulate them in such a way that they refer to those sources that we control i.e. "free" charge and currents.

here $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

$\vec{H} = \dfrac{\vec{B}}{\mu_0} - \vec{M}$

For linear media, $\vec{D} = \varepsilon \vec{E}$ $\qquad \varepsilon = \varepsilon_0 (1+\chi_e)$

$\vec{H} = \dfrac{\vec{B}}{\mu}$ $\qquad \mu = \mu_0 (1+\chi_m)$

Boundary Conditions

1. $\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f$
2. $\vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = 0$ $\qquad (\vec{\nabla}\times\vec{D} \neq 0$ but $\vec{\nabla}\times\vec{E} = 0)$
3. $B_1^{\perp} - B_2^{\perp} = 0$
4. $\dfrac{1}{\mu_1}\vec{B}_1^{\parallel} - \dfrac{1}{\mu_2}\vec{B}_2^{\parallel} = \vec{k}_f \times \hat{n}$ $\qquad (\vec{\nabla}\times\vec{H} = J_{free})$

जिन surfaces के across field component continuous हैं, उनमें तो normal field ($E_{\parallel}$, $B_{\perp}$) आएँगी !!
जिन components में discontinuity हैं, उनमें modified forms आएँगी ($D_{\perp}$, $H_{\parallel}$)

In electrostatics & magnetostatic, $\rho_b = -\vec{\nabla}\cdot\vec{P}$; $J_b = \vec{\nabla}\times\vec{M}$

In electrodynamics cases, there is 1 more thing : J_p Polarization current density

Any change in Polarization involves flow of bound charges
suppose Polarization introduces a charge density $\sigma_b = P$ at one end and $-\sigma_b$ at other. If P now increases a bit, charge on each end increases $\Rightarrow$ new current $\underline{\vec{J}_p = \left(\dfrac{\partial \vec{P}}{\partial t}\right)}$

$\Rightarrow \vec{J} = J_f + J_b + J_p = J_f + \vec{\nabla}\times\vec{M} + \left(\dfrac{\partial \vec{P}}{\partial t}\right)$

$\Rightarrow \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}_f + \mu_0(\vec{\nabla}\times\vec{M}) + \mu_0\left(\dfrac{\partial \vec{P}}{\partial t}\right) + \mu_0 \dfrac{\partial}{\partial t}\vec{D} - \mu_0\left(\dfrac{\partial \vec{P}}{\partial t}\right) \Rightarrow \vec{\nabla}\times\vec{H} = J_f + (\partial \vec{D}/\partial t)$

E & M (9) 23/01/2012

- Biot Savert's law : $d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \vec{r}}{r^3}$

application : ① long wire
② circular coil
③ solenoid

Maxwell Modification of Ampere's law

- Ampere's law $\oint_s \vec{B} \cdot d\vec{l} = \mu_0 I$

- Conservation of charge

$$I = \int_s \vec{J} \cdot d\vec{s}$$

$$\vec{I} \cdot d\vec{l} = \int_V \vec{J} \, d\tau$$

$$I = -\frac{dQ}{dt}$$

$$\int_s \vec{J} \cdot d\vec{s} = -\frac{d}{dt} \int_V \rho \, dv$$

$$\int \nabla \cdot J \, dV = -\int \frac{d\rho}{dt} \, dV$$

$$\Rightarrow \int \left(\nabla \cdot J + \frac{\partial \rho}{\partial t} \right) d\tau = 0$$

$$\Rightarrow \boxed{\nabla \cdot J + \left(\frac{\partial \rho}{\partial t} \right) = 0}$$

always valid "Law of Conservation of Charge"

We know by Ampere's law,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \int_s \vec{J}\cdot d\vec{s}$$

$$\int (\vec{\nabla}\times\vec{B})\, d\vec{s} - \int \mu_0 J \cdot d\vec{s} = 0$$

$$\Rightarrow \int (\vec{\nabla}\times\vec{B} - \mu_0 \vec{J})\, d\vec{s} = 0$$

$$\Rightarrow \boxed{\frac{1}{\mu_0}(\vec{\nabla}\times\vec{B}) = \vec{J}}$$

But divergence of a curl = 0

$$\Rightarrow \nabla\cdot\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$$

But by continuity of charge, $\nabla\cdot\vec{J} = -\left(\frac{\partial\rho}{\partial t}\right)$

We know $\rho = \epsilon_0 \vec{\nabla}\cdot\vec{E}$

$$\frac{d\rho}{dt} = \vec{\nabla}\cdot\left(\epsilon_0 \frac{\partial\vec{E}}{\partial t}\right)$$

Conservation of charge cannot be violated.
Hence Ampere's law is valid in form of $\left[\oint \vec{B}\cdot d\vec{l} = \mu_0 I\right]$ only when $\left(\frac{\partial\vec{E}}{\partial t}\right) = 0$ i.e. Electric Field is not varying

Applying the correction for time varying fields,

$$\boxed{\vec{J} = \sigma\vec{E}}$$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J} + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right]}$$

This correction to Ampere's law was done by Maxwell for time varying Electric Fields.

We know for.

$$\boxed{\vec{P} + \epsilon_0 \vec{E} = \vec{D}} \; ; \; \boxed{\frac{\vec{B}}{\mu_0} - \vec{M} = \vec{H}} \; ; \; \begin{array}{l} J_b = \vec{\nabla} \times \vec{M} \\ J_p = \left(\frac{\partial \vec{P}}{\partial t}\right) \end{array}$$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{H} = \left[\vec{J}_f + \frac{\partial \vec{D}}{\partial t} \right]}$$

Define $\frac{\partial \vec{D}}{\partial t}$ as surface Displacement current density, J_D.

We know, $\vec{J}$ is called surface conduction current density, J.

Displacement current arises due to variation of charge density.

$$J_D = (\partial D / \partial t)$$

$$I_D = J_D \cdot A$$

$$\boxed{I_D = \epsilon A \left(\frac{\partial E}{\partial t}\right)}$$

Experimental foundation of Amperes law is in Biot Savart's law

○ Faraday's law of Electromagnetic Induction

$$\text{Flux } \Phi_B = \int_S \vec{B} \cdot d\vec{s} = \vec{B} \cdot \vec{A}$$

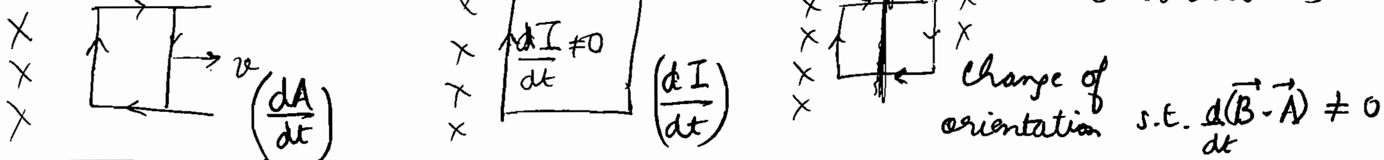
∗ Note that $\int_S \vec{B} \cdot d\vec{a} \neq 0$ but rather $\oint_S B \cdot d\vec{a} = 0$

It can be hypothetically represented as lines of force.

$$\boxed{\frac{d\phi}{dt} = \vec{B} \cdot \left(\frac{d\vec{A}}{dt}\right) + \left(\frac{d\vec{B}}{dt}\right) \cdot \vec{A}}$$

We know $B \propto I \Rightarrow \left(\frac{d\vec{B}}{dt}\right) \propto \left(\frac{dI}{dt}\right)$

3 variations



$\phi_B = BA\cos\omega t$ Rotation of current loop.

[For n turns of coil,
$\phi_B = nBA\cos\omega t$]

$$\frac{d\phi_B}{dt} = -\omega BA \sin(\omega t)$$

This was observed !!

Due to any of these 3 changes, whenever ϕ changes, there is an induced emf. The direction of flow of current can be understood from Lenz's law.

$$\boxed{\mathcal{E}_{emf} = \int_c \vec{E}\cdot d\vec{l} = -\left(\frac{d\phi_B}{dt}\right)}$$ Faraday's Law

Now we need to write the law in integral & differential forms

$$\oint_l \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_s \vec{B}\cdot d\vec{s}$$

Note that here we are interested in a closed loop & not closed surface ⇒ we will apply Stokes Law

$$\boxed{\oint_l \vec{E}\cdot d\vec{l} = -\int_s \left(\frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{s}}$$ Faraday's law in Integral form

$$\int (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = -\int \left(\frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{s}$$

$$\Rightarrow \int \left[\vec{\nabla}\times\vec{E} + \left(\frac{\partial \vec{B}}{\partial t}\right)\right]\cdot d\vec{s} = 0$$

$$\Rightarrow \boxed{\nabla\times\vec{E} + \left(\frac{\partial \vec{B}}{\partial t}\right) = 0}$$ Faraday's law in differential form

For electrostatic field $\vec{\nabla} \times \vec{E} = 0$

but for time varying fields, $\vec{\nabla} \times \vec{E} = -\left(\frac{\partial B}{\partial t}\right)$

It is an experimental law.

⊙ Symmetry in Nature and Faraday's law motivated Maxwell to introduce correction into Ampere's law.

Maxwell Equations

From the theory studied so far,

$$\left[\oint_S \vec{E} \cdot d\vec{s} = \frac{q_t}{\epsilon_0} = \frac{1}{\epsilon_0} \int_V \rho_t \, dV\right]$$ Gauss Law in Integral Form

$$\left[\begin{array}{ll} \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} & : \text{Free Space} \\ \nabla \cdot \vec{E} = \frac{\rho_f}{\epsilon} & : \text{linear dielectric Material} \end{array}\right]$$ Differential Form of Gauss law

$$\left[\begin{array}{l} \vec{\nabla} \cdot \vec{B} = 0 \\ \oint_S \vec{B} \cdot d\vec{s} = 0 \end{array}\right]$$ Monopoles do not exist

$$\left[\begin{array}{l} \int_C \vec{E} \cdot d\vec{l} = -\int_S \left(\frac{\partial \vec{B}}{\partial t}\right) \cdot d\vec{s} \\ \vec{\nabla} \times \vec{E} = -\left(\frac{\partial B}{\partial t}\right) \end{array}\right]$$ Faraday's law of Electromagnetic Induction

$$\left[\begin{array}{l} \int_C \vec{B} \cdot d\vec{l} = \mu_0 \int (\vec{J} + \vec{J_d}) \, d\vec{s} \\ \qquad = \mu_0 (I + I_d) \\ (\vec{\nabla} \times \vec{B}) = \mu_0 [J + J_d] \end{array}\right]$$ Ampere's law as Corrected by Maxwell $\vec{J_d} = \epsilon\left(\frac{\partial \vec{E}}{\partial t}\right)$

Maxwell's Equations in Electromagnetics

$$\vec{\nabla}\cdot\vec{E} = \left(\frac{\rho}{\epsilon_0}\right) \quad \text{— (1)} \checkmark$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{— (2)} \checkmark$$

$$\vec{\nabla}\times\vec{E} = -\left(\frac{\partial B}{\partial t}\right) \quad \text{— (3)} \checkmark$$

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \vec{J}_D\right) \quad \text{— (4)} \checkmark$$

Note that differential form of Maxwell Equations is always preferred.

There are 3 cases of interest :-

1) Free Space (Vacuum)

2) Dielectrics : Isotropic or Anisotropic

dielectric are classified according to Permittivity ϵ

If ϵ is same everywhere : ISOTROPIC

If ϵ is changing : ANISOTROPIC

$$\checkmark\ \epsilon = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{xy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

3) Conducting Medium

Force free space,

$$\vec{J}_e = \sigma \vec{E}$$

$$\vec{J}_D = \left(\frac{\partial \vec{D}}{\partial t}\right)$$

Note that variable in Maxwell Equations are

$$\rho, \underset{\text{Conductivity}}{\sigma}, \epsilon, \mu$$

For free space, $\rho = 0$ ✓

$\sigma = 0$ ✓

$\epsilon = \epsilon_0$ ✓

$\mu = \mu_0$ ✓

$\Rightarrow$ $\vec{\nabla} \cdot \vec{E} = 0$ ✓

$\vec{\nabla} \cdot \vec{B} = 0$ ✓

$\vec{\nabla} \times \vec{E} = -\left(\frac{\partial \vec{B}}{\partial t}\right)$ ✓

$\vec{\nabla} \times \vec{B} = +\mu_0 J_D = +\mu_0 \epsilon_0 \left(\frac{\partial \vec{E}}{\partial t}\right)$ ✓

Note that its a flawed reasoning that since vacuum के लिए ϵ_0 $\Rightarrow$ material के लिए ϵ.

Actually for material, we have addition polarization current

$\Rightarrow \frac{\partial}{\partial t}(\epsilon_0 E + P) = \frac{\partial D}{\partial t} = \frac{\partial \epsilon E}{\partial t}$

$= \epsilon \left(\frac{\partial E}{\partial t}\right)$

For isotropic dielectric,

$\epsilon = \epsilon$

$\mu = \mu_0$ [Non Magnetic dielectric]

$\rho = 0$ [No charge in glass]

$\sigma = 0$ [Dielectric is non conducting medium]

$\Rightarrow$ $\vec{\nabla} \cdot \vec{E} = 0$ ✓, $\vec{\nabla} \cdot \vec{B} = 0$ ✓, $\vec{\nabla} \times \vec{E} = -\left(\frac{\partial \vec{B}}{\partial t}\right)$ ✓, $\vec{\nabla} \times \vec{B} = \mu_0 \epsilon \left(\frac{\partial \vec{E}}{\partial t}\right)$ ✓

For Anisotropic dielectric, (eg. Crystals are anisotropic different behaviour in preferred directions)

$\rho = 0, \sigma = 0$

$$\vec{D} = \epsilon \vec{E} \Rightarrow \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

for $\vec{\nabla} \cdot \vec{D} = 0$ ✓

i.e. $\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0$

① $\Rightarrow \frac{\partial}{\partial x}[\epsilon_{xx} E_x + \epsilon_{xy} E_y + \epsilon_{xz} E_z] + \frac{\partial}{\partial y}[\quad] + \frac{\partial}{\partial z}[\quad] = 0$

It can be seen that for vector form, equation is simple. But for component wise equation is lengthy.

② $\vec{\nabla} \cdot \vec{B} = 0$ ✓

③ $\vec{\nabla} \times \vec{E} = -\left(\frac{\partial B}{\partial t}\right)$ ✓

④ $\vec{\nabla} \times \vec{B} = \mu_0 \left(\frac{\partial \vec{D}}{\partial t}\right)$ ✓

For Conducting Media,

$\rho = 0, \quad \sigma = \sigma, \quad \epsilon = \epsilon, \quad \mu = \mu_0$ [Non Magnetic]

[const.] if isotropic Conducting Media

$\vec{\nabla} \cdot \vec{E} = 0$ ✓

$\vec{\nabla} \cdot \vec{B} = 0$ ✓

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ✓

$\vec{\nabla} \times \vec{B} = \mu_0 \sigma \vec{E} + \mu_0 \epsilon \left(\frac{\partial \vec{E}}{\partial t}\right)$ ✓

The next step in all Maxwell Equations is to find out $\vec{E}$ and $\vec{B}$. The solution is wave equations $\nabla^2\psi = \frac{1}{v_p^2}\left(\frac{\partial^2\psi}{\partial t^2}\right)$

where ψ comes out to be $\boxed{\psi = \psi_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}}$

s.t. $v_p = \left(\frac{\omega}{k}\right)$

TAKE THE CURL OF 3rd & 4th

Curl of $\vec{\nabla} \times \vec{E}$

For Free space,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \times \left(\frac{\partial \vec{B}}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \vec{E}(\vec{\nabla}\cdot\vec{\nabla}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

Invoking ① and ④ equations,

$$\vec{\nabla}(0) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$\boxed{\nabla^2\vec{E} = \mu_0\epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}}$$

For dielectric isotropic

(Note that $\rho = 0$ for dielectric

$$\boxed{\nabla^2\vec{E} = \mu\epsilon\left(\frac{\partial^2 E}{\partial t^2}\right)}$$

For conducting isotropic

$$\boxed{\nabla^2\vec{E} = \mu\sigma\left(\frac{\partial \vec{E}}{\partial t}\right) + \mu\epsilon\left(\frac{\partial^2 \vec{E}}{\partial t^2}\right)}$$

Curl of $\vec{\nabla} \times \vec{B}$

For Free space,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \times \left[\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right]$$

$$\nabla^2 B = -\mu_0\epsilon_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E}) = -\mu_0\epsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial B}{\partial t}\right) = \mu_0\epsilon_0\left(\frac{\partial^2 B}{\partial t^2}\right)$$

$$\Rightarrow \boxed{\nabla^2 B = \mu_0\epsilon_0 (\partial^2 B/\partial t^2)}$$

Conducting Media

$$\boxed{\nabla^2 \vec{B} = \mu_0 \sigma \left(\frac{\partial \vec{B}}{\partial t}\right) + \mu_0 \epsilon \left(\frac{\partial^2 \vec{B}}{\partial t^2}\right)}$$

Since in all media, $\vec{E}$ and $\vec{B}$ follow same equations, hence both are always in same phase.

Define Mathematical Operator

$$\Box^2 = \left[\nabla^2 - \cancel{\mu_0}\epsilon \frac{\partial^2}{\partial t^2}\right] \quad : \quad \text{de Alembertian Operator}$$

for dielectric

$$\left[\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2}\right]\{\vec{E}\} = 0 \quad \& \quad \left[\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2}\right]\{\vec{B}\} = 0$$

$$\boxed{\Box^2 = \nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}} \quad \text{where} \quad v = \frac{1}{\sqrt{\mu\epsilon}}$$

For free space,

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \left(\frac{\partial^2 \vec{E}}{\partial t^2}\right)$$

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \left(\frac{\partial^2 \vec{B}}{\partial t^2}\right)$$

these are wave equations $\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{4\pi \times 10^{-7} \times \frac{1}{9 \times 4\pi \times 10^9}}} = 3 \times 10^8 \text{ m/s}$$

For any dielectric, $v = \frac{1}{\sqrt{\mu_0 \epsilon_0 \epsilon_r}} = \frac{c}{\sqrt{\epsilon_r}} = \left(\frac{c}{n}\right)$

refractive index

$$\Rightarrow \boxed{n = \sqrt{\epsilon_r}}$$

[although not explained empirically]

For most of the materials $\mu \simeq \mu_0$ $\therefore$ not much contribution to n

We know,

$$\vec{P} = (\epsilon_r - 1)\,\epsilon_0 \vec{E} \quad \text{(for linear dielectric)}$$

$$\epsilon_r = \frac{\vec{P}}{\epsilon_0 \vec{E}} + 1$$

$$\Rightarrow \boxed{n^2 = \frac{|\vec{P}|}{|\epsilon_0 \vec{E}|} + 1}$$

$$\psi = \psi_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\boxed{\begin{aligned} \vec{E}(r,t) &= \vec{E_0}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \\ \vec{B}(r,t) &= \vec{B_0}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \end{aligned}}$$

✓ PHASE PART OF BOTH $\vec{E}$ and $\vec{B}$ are same .....

$$\boxed{v_p = \left(\frac{\omega}{k}\right) = \frac{1}{\sqrt{\mu\epsilon}}}$$

✓ Now EM waves are produced only when $\vec{E}$ and $\vec{B}$ change w.r.t. time. It happens when charges are accelarated or decelarated. It leads to propagation of $\vec{E}$ and $\vec{B}$ in space. Accelaration / Deceleration usually occurs whenever charge particles hit some surface & collision occurs.

E&M (10) 24/01/2012

- Note that the 2 $\vec{E}$ & $\vec{B}$ are in same phase as long as $\vec{k}$ is real.

$$\left(\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2}\right)\vec{\psi} = \mu\epsilon \frac{\partial \vec{\psi}}{\partial t}$$

$\vec{k}$ is imaginary for conductors.

Conductor
- attenuation
- $\Delta\phi$ between E and B
- k imaginary

$$\psi = \psi_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

For conductors, we write $\vec{k}^*$

$$\vec{k}^* = \vec{k}_+ + i\,\vec{k}_-$$

$$\Rightarrow \psi = \psi_0 \, e^{-i\omega t}\left[e^{i\vec{k}_+\cdot\vec{r}} \, e^{i i \vec{k}_-\cdot\vec{r}}\right]$$

$$= \psi_0 \, e^{-\vec{k}_-\cdot\vec{r}}\left[e^{i(\vec{k}_+\cdot\vec{r} - \omega t)}\right]$$

Conductor

It is attenuated wave equation.

If $\sigma = 0$, then it is dielectric $\Rightarrow$ k is real

$\Rightarrow$ No attenuation of Amplitude

$$\psi = \psi_0 \left[e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right]$$

Dielectric

Note that, for conductors, we can also write

$$\vec{k} = \vec{\alpha} + i\vec{\beta}$$

$\vec{\alpha}$: Phase Factor; $\vec{\beta}$: Attenuation Factor

$$\Rightarrow \psi = \psi_0 \, e^{-\beta r} \, e^{-i(\vec{\alpha}\cdot\vec{r} - \omega t)}$$

$$v_p = \left(\frac{\omega}{\alpha}\right)$$

Proof of Transverse Nature of EM Waves :-

$$\nabla^2 \psi = \mu \epsilon \left(\frac{\partial^2 \psi}{\partial t^2} \right)$$

* In order to prove transverse nature, we will need to define [redacted] operators corresponding to $\vec{\nabla}$ and $\frac{\partial}{\partial t}$

Solution for this equation :

$$\psi = \psi_0 \, e^{-i(\vec{k}\cdot\vec{x} - \omega t)}$$

Differentiating,

$$\left(\frac{\partial \psi}{\partial t} \right) = \psi_0 \, e^{-i(\vec{k}\cdot\vec{x} - \omega t)} \quad i\omega$$

$$\Rightarrow \frac{\partial(\psi)}{\partial t} = i\omega(\psi)$$

<Using it as Operator>

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \Leftrightarrow i\omega} \quad \& \quad \boxed{\omega \Leftrightarrow -i\frac{\partial}{\partial t}}$$

$$\vec{\nabla}\psi = \sum i\frac{\partial}{\partial x}\left[\psi_0 \, e^{i\omega t} \, e^{-i(k_x x + k_y y + k_z z)} \right]$$

$$= \psi_0 \, e^{i\omega t}\left[\sum \hat{i} \; e^{-i(\vec{k}\cdot\vec{x})} \cdot -ik_x \right]$$

$$= -i\vec{k}\,\psi$$

$$\Rightarrow \vec{\nabla}(\psi) = -i\vec{k}(\psi)$$

$$\Rightarrow \boxed{\vec{\nabla} \Leftrightarrow -i\vec{k}} \quad \& \quad \boxed{\vec{k} \Leftrightarrow i\vec{\nabla}}$$

Using the operators, recasting the Maxwell equations,

$$\vec{\nabla}\cdot\vec{E} = 0 \quad \Rightarrow \quad \vec{k}\cdot\vec{E} = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad\qquad \vec{k}\cdot\vec{B} = 0$$

transverse

$$\vec{\nabla}\times\vec{E} = -i\omega B \qquad -i\vec{k}\times\vec{E} = -i\omega B$$

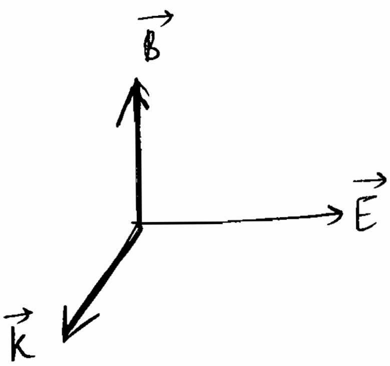
$$\Rightarrow \boxed{\vec{k}\times\vec{E} = \omega\vec{B}}$$

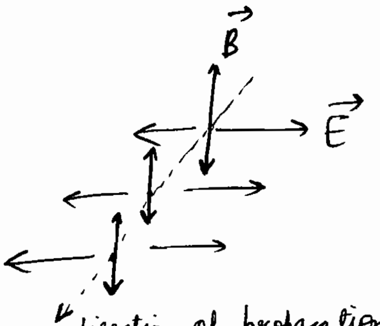
$$\vec{\nabla}\times\vec{B} = \mu\epsilon\frac{\partial\vec{E}}{\partial t} \quad \Rightarrow \quad -i\vec{k}\times\vec{B} = \mu\epsilon\, i\omega\vec{E}$$

$$\Rightarrow \boxed{\vec{k}\times\vec{B} = -\mu\epsilon\omega\vec{E}}$$

Righthand triad of $[\vec{E},\vec{B},\vec{k}]$

All the equations prove that all 3 are perpendicular and $[\vec{E},\vec{B},\vec{k}]$ form a right handed triad





Till now we have seen 4 properties:

- $v = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c$ in empty space
- $v = \frac{c}{n} = \frac{c}{\sqrt{k}}$ in medium
- waves are transverse in nature
- $[\vec{E},\vec{B},\vec{k}]$ form a right handed triad.
- $\vec{E} = v\vec{B}$

$$\vec{B} = \left(\frac{\vec{k} \times \vec{E}}{\omega}\right)$$

From 3rd equation :

$$\boxed{\frac{B_0}{E_0} = \frac{k}{\omega}}$$

k can be real or complex

Taking the Amplitudes,

$$B = \frac{kE}{\omega} = \left(\frac{E}{v}\right)$$

$$\boxed{\frac{E}{B} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} \text{ m/s}}$$

$$\frac{E}{H} = Z = \frac{Z_0}{n}$$

$$\boxed{\frac{E}{H} = \frac{\mu}{\sqrt{\mu\epsilon}} = \sqrt{\frac{\mu}{\epsilon}} \text{ ohms}}$$

$$\Rightarrow \boxed{H = \frac{nE}{Z_0}}$$

For free space, $\dfrac{E}{H} = \sqrt{4\pi \times 10^{-7} \times 4\pi \times 9 \times 10^{9}}$ Ohms $= 4\pi \times 3 \times 10\ \Omega$

$= 120\pi\Omega = 376.9\Omega$

$\frac{E}{H}$ is the impedance of electromagnetic waves in the medium.

It is the 'characteristic impedance' of the medium.

$\sqrt{\frac{\mu_0}{\epsilon_0}}$ is the characteristic impedance (Z_0) of free space $= 377\Omega$

$$Z = \frac{|\vec{E}|}{|\vec{H}|} = \sqrt{\frac{\mu}{\epsilon}} = \underbrace{\sqrt{\frac{\mu_0}{\epsilon_0\epsilon_r}}}_{\text{in a dielectric}} = \frac{Z_0}{\sqrt{\epsilon_r}} = \left(\frac{Z_0}{n}\right)$$

(*) It is the resistance offered by the medium for the propagation of EM wave.

→ 4th equation is used only for verification purpose.

Poynting's Theorem

We know

$$\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega}$$

$$\Rightarrow \quad \vec{E} \times \vec{B} = \vec{E} \times \frac{(\vec{k} \times \vec{E})}{\omega}$$

$$= \frac{1}{\omega}\left[\vec{k}(\vec{E}.\vec{E}) - \cancel{\vec{E}(\vec{E}.\vec{k})}\right]$$

$$= \frac{\vec{k}}{\omega} E^2 = \left(\frac{E^2}{\omega}\right)\vec{k}$$

$$\Rightarrow \quad \boxed{\vec{E} \times \vec{B} = \frac{E^2}{\omega}\vec{k}}$$

* 2 things to note in the proof :

① $\int_v \nabla \cdot (\vec{E} \times \vec{H})\, d\tau = \int (\vec{E} \times \vec{H}) \cdot d\vec{a}$

② $\nabla \cdot (\vec{A} \times \vec{B}) = (\vec{\nabla} \times \vec{A}) \cdot \vec{B} - (\vec{\nabla} \times \vec{B}) \cdot \vec{A}$

(note the negative sign)

$$\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{B})$$

Note that divergence of a curl = 0 but divergence of cross product may not be zero

Similarly, dividing by μ

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E}(\vec{\nabla} \times \vec{H})$$

From 4th eqn, $\vec{E} \cdot \vec{\nabla} \times \vec{H} = \left[\vec{J} + \left(\frac{\partial \vec{D}}{\partial t}\right)\right] \cdot \vec{E}$ — ①

From 3rd eqn, $\vec{H} \cdot \vec{\nabla} \times \vec{E} = -\mu\left(\frac{\partial \vec{H}}{\partial t}\right) \cdot \vec{H}$ — ②

Subtracting ① from ②

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = -\mu \vec{H} \cdot \left(\frac{\partial \vec{H}}{\partial t}\right) - \vec{E} \cdot \left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right]$$

$$= -\vec{E} \cdot \vec{J} - \frac{\partial}{\partial t}\left[\frac{1}{2}\mu(\vec{H} \cdot \vec{H}) + \frac{1}{2}\epsilon(\vec{E} \cdot \vec{E})\right]$$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\vec{E}\cdot\vec{J} - \frac{\partial}{\partial t}\left[\frac{1}{2}\mu H^2 + \frac{1}{2}\epsilon E^2\right]$$

Taking the volume integral on both sides,

$$\int_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\, dV = -\int_V \vec{E}\cdot\vec{J}\, dV - \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\mu H^2 + \frac{1}{2}\epsilon E^2\right) dV$$

$$\boxed{-\int (\vec{E}\times\vec{H})\cdot d\vec{A} = +\int_V \vec{E}\cdot\vec{J}\, dV + \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\mu H^2 + \frac{1}{2}\epsilon E^2\right) dV}$$

Energy Continuity Equation i.e. Statement of Conservation of Energy in Electromagnetic wave.

$$\Rightarrow \int_V \vec{E}\cdot\vec{J}\, dv + \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\mu H^2 + \frac{1}{2}\epsilon E^2\right) dV + \int (\vec{E}\times\vec{H})\cdot d\vec{A} = 0$$

↑ Work done by EM wave on field charge

↑ Total electrical & Magnetic Energy stored with EM field

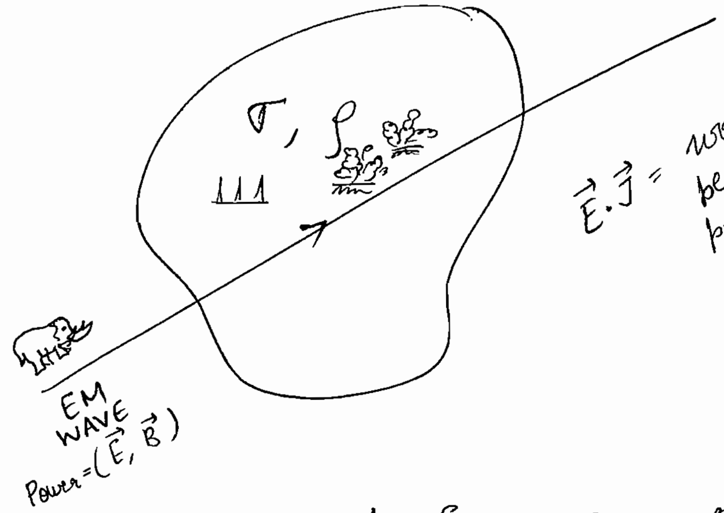
↑ Energy carried out by the wave

define: $\vec{S} = \vec{E}\times\vec{H}$: Energy carried out per second per unit area in the direction of propagation of wave !!

POYNTING VECTOR

In wave theory, its called Intensity

✔ Note that the Poynting Equation is very general



$\vec{E}\cdot\vec{J} =$ work done per unit time per unit volume

- $\int_v \vec{J}\cdot\vec{E}\, dv$: Work done by EM wave on charges

If $\sigma = 0 \Rightarrow \vec{J} = \sigma\vec{E} = 0 \Rightarrow$ NO Work done.

$$\vec{F} = \left[q\vec{E} + q(\vec{v}\times\vec{B}) \right]$$ (Force exerted by Elephant or EM Wave)

~~...~~, Work $= \vec{F}\cdot\vec{v}dt = \int q\vec{E}\cdot\vec{v}dt$

$$\Rightarrow \left(\frac{dW}{dt}\right) = \int_V \rho\, d\tau\, \vec{E}\cdot\vec{v}$$

$$= \int_V \vec{E}\cdot\vec{J}\, d\tau$$

$\boxed{q = \rho\, d\tau}$

$\boxed{\rho\vec{v} = \vec{J}}$

$\boxed{\sigma\vec{E} = \vec{J}}$ (Ohmic materials)

This term becomes 0 when either $\sigma = 0$ or $\rho = 0$

- Energy associated with EM wave having $\vec{E}, \vec{B}$

$$= \int_V \left(\frac{1}{2}\mu H^2 + \frac{1}{2}\epsilon E^2\right) dV$$

Loss of energy due to resistance by छोटे and बड़े and थकावट

$$= \frac{\partial}{\partial t}\left[\int_V \frac{1}{2}\mu H^2 + \frac{1}{2}\epsilon E^2\right]$$

- After doing work and encountering loss of energy, the elephant carries $(\vec{E} \times \vec{H})$ energy per unit area per unit time.

- Electrical Energy : Magnetic Energy

$$\frac{1}{2} \epsilon E^2 : \frac{1}{2} \mu H^2 \quad \text{(per unit volume)}$$

$$= \frac{1}{2} \frac{\epsilon}{\mu} : \frac{1}{2} \frac{\epsilon}{\mu}$$

$$= 1 : 1$$

$$\Rightarrow \boxed{\frac{1}{2} \epsilon E^2 = \frac{1}{2} \mu H^2}$$ Electrical energy = Magnetic Energy

$$\Rightarrow \frac{\partial}{\partial t} \int_V \frac{1}{2} \mu H^2 + \frac{1}{2} \epsilon E^2 \, dV$$

$$= \frac{\partial}{\partial t} \int_V \mu H^2 \, dV = \frac{\partial}{\partial t} \int_V \epsilon E^2 \, dV$$

If $\sigma = 0$,

$$\int \vec{S} \cdot d\vec{A} = -\frac{\partial}{\partial t} \int \epsilon E^2 \, dV = -\frac{\partial}{\partial t} \int U \, dV$$

* U: energy per unit volume $= \varepsilon E^2$

$$\Rightarrow \int_V \vec{\nabla} \cdot \vec{S} \, dV = \int -\frac{\partial U}{\partial t} \, dV$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{S} + \frac{\partial U}{\partial t} = 0}$$ ← that why called Energy Continuity Theorem

$$\vec{S} = \vec{E} \times \vec{H}$$

$$= \frac{1}{\mu} (\vec{E} \times \vec{B})$$

$$\boxed{\vec{S} = \frac{1}{\mu} \frac{E^2}{\omega} \vec{k}}$$

$$|\vec{S}| = \frac{E^2 |\vec{k}|}{\mu \omega} = \frac{E^2 \sqrt{\mu \epsilon}}{\mu} = \left[\frac{E^2}{Z}\right]$$

$$\boxed{|\vec{S}| = \frac{E^2}{Z}} \boxed{= H^2 Z}$$

For free space

$$|\vec{S}| = \left(\frac{E^2}{Z_0}\right) = E^2 \sqrt{\frac{\epsilon_0}{\mu_0}} = \epsilon_0 c E^2$$

$$\boxed{|\vec{S}| = \epsilon_0 c E^2}$$ instantaneous

$$\Rightarrow \boxed{|\vec{S}| = c U}$$

$$\boxed{\text{total energy per unit volume} = \epsilon E^2}$$

We know,

$$E = E_0 e^{-i(\vec{k}\cdot\vec{r} - \omega t)}$$

We can write, $E = E_0 \sin(\vec{k}\cdot\vec{r} - \omega t)$ or $E = E_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$

$$\Rightarrow |\vec{S}|_{avg} = \epsilon_0 c \, E_0^2 \langle \sin^2(\vec{k}\cdot\vec{r} - \omega t) \rangle$$

$$\boxed{|\vec{S}|_{avg} = \frac{1}{2} c \epsilon_0 E_0^2}$$ average

$$\Rightarrow \boxed{|\vec{S}|_{avg} = c \, U_{avg}}$$

Continuity Equation

What precisely does conservation of charge means:

(1) Global Conservation : Total charge in the Universe is const.

(2) Local Conservation : If the total charge in some volume changes, then exactly that amount of charge must have passed in or passed out of the volume through the surface.

i.e. $\frac{dQ}{dt} = -\oint_S \vec{J} \cdot d\vec{a}$

$$\Rightarrow \int_V \left(\frac{\partial \rho}{\partial t}\right) d\tau = -\int_V (\vec{\nabla} \cdot \vec{J}) \, . \, d\tau$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{J} = -\left(\frac{\partial \rho}{\partial t}\right)}$$

Note that it is not an independent assumption but a consequence of the laws of electrodynamics.

i.e. From 4^{th} eqn: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \left(\frac{\partial \vec{E}}{\partial t}\right)$

Taking divergence,

$$0 = \mu_0 . \vec{\nabla} \cdot \vec{J} + \mu_0 \epsilon_0 . \frac{\partial}{\partial t} . \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{J} = -\left(\frac{\partial \rho}{\partial t}\right)}$$

Poynting's Theorem

Its the energy conservation theorem of electrodynamics
It states that work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the fields, less the energy that flowed out through the surface.

$$\int E\cdot J \, d\tau = -\frac{\partial}{\partial t}\int\left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0}\right) d\tau - \oint \frac{E\times B}{\mu}\, d\vec{a}$$

For Understanding

Similar to 1^{st} law of Thermodynamics

$$Q = dU + PdV$$

$\int_V \vec{E}\cdot\vec{J}\, d\tau$

(work done)

$-\oint_S E\times H \, d\vec{a}$

(energy coming in)

$$+\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right) d\tau$$

(change in energy)

Electromagnetic Waves

General Wave

A wave is a disturbance of a continuous medium that propagates with a fixed shape at constant velocity.

Mathematically, the equation for a wave is

$$f(x,t) = f(x - vt, 0) = \psi(x - vt)$$

For a wave function that will satisfy the below relation can be represented as a wave, as it can be represented in a relation $\psi(x-vt)$.

$$\left(\frac{\partial^2 \psi}{\partial x^2}\right) = \frac{1}{v^2}\left(\frac{\partial^2 \psi}{\partial t^2}\right)$$

i.e. ψ should be a function that depends on (x,t) in the special combination $(x - vt)$

But its nicer to write the waves in terms of ω rather than v, in the form

$$\psi(kx-\omega t)$$

where $k = \frac{2\pi}{\lambda}$: wave number

$\left(\frac{\omega}{k}\right) = v$: velocity of wave

ω : angular frequency of the wave

Complex Notation of a general wave

Now one possible solution of wave is

$$\psi(x,t) = A\cos(kx-\omega t+\delta)$$

Also from fourier analysis we know that any wave can be linear as linear combination of sinusoidal waves, and therefore, it is sufficient for us to analyze just sinusoidal waves.

Now we know exponentials are much easier to manipulate than sines & cosines.

$\therefore$ introducing $\tilde{\psi}(x,t) = \tilde{A}\, e^{i(kx-\omega t)}$: Complex Wave Function

where $\tilde{A} = A e^{i\delta}$: complex Amplitude

$$\psi(x,t) = \text{Re}\,[\tilde{\psi}(x,t)]$$

The actual wave function is the real part of complex wave function.

Remember that complex exponentials are simply for mathematical ease.

Real wave is always real part of the complex wave.

Some more points on field inside conductors

As we know waves (electric field) Cannot reside inside a Conductor, we will have have an attenuated wave. where $\vec{k}$ is ~~imaginary~~ complex

$$\tilde{k} = \alpha + i\beta$$

$$\alpha = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1\right]^{1/2}$$

$$\beta = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1\right]^{1/2}$$

$$\tilde{E} = \tilde{E}_0\, e^{-\beta x}\, e^{i(\alpha x - \omega t)}$$

$$\tilde{B} = \tilde{B}_0\, e^{-\beta x}\, e^{i(\alpha x - \omega t)}$$

From Maxwell equations

$$\boxed{\tilde{B}_0 = \frac{|\tilde{k}|}{\omega}\,\tilde{E}_0}$$

Now since k is complex, B and E are not longer in same phase.

Write $k = \sqrt{\alpha^2+\beta^2}\ \angle \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

$$\frac{B_0}{E_0} = \frac{|\tilde{k}|}{\omega} = \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}$$

Note that we have $e^{i(\alpha x-\omega t)}$

$\therefore$ for additional $e^{i\theta}$ where $\theta = \tan^{-1}(\beta/\alpha)$, we have

$e^{i(\alpha x - \omega t + \theta)}$

$= e^{i(\alpha x - \omega(t - \frac{\theta}{\omega}))}$

$=$ LAG by $t_0 = \left(\frac{\theta}{\omega}\right)$ i.e. $\phi = \theta$

Magnetic field lags electric fied by $\tan^{-1}\left(\frac{\beta}{\alpha}\right)$

eg. for good conductors $(\alpha = \beta)$, magnetic field lags the electric field by 45°. For any conductor, lag is something in between 0° to 45°.

Q16)

$$\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega}$$

$$= \frac{k\hat{z}}{\omega} \times E_0 \cos(kz - \omega t + \delta)\,\hat{x}$$

$$= \frac{kE_0}{\omega} \cos(kz - \omega t + \delta)\,\hat{y}$$

$$U = \frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2$$

$$= \epsilon E^2$$

$$= \epsilon^2 E_0^2 \cos^2(kz - \omega t + \delta)$$

$$S = vU$$

$$= \frac{\omega}{k} \cdot \epsilon^2 E_0^2 \cos^2(kz - \omega t + \delta)$$

$$\langle S \rangle = \frac{1}{2} \frac{\omega}{k} \epsilon^2 E_0^2$$

Q17) Assuming Area = Circle = πR^2

$$\therefore S = \left(\frac{\text{Power}}{\text{Area}}\right)$$

$$\langle S \rangle = \frac{1}{2} \frac{E_0^2}{Z_0} \Rightarrow E_0 = \sqrt{2Z_0 \langle S \rangle}$$

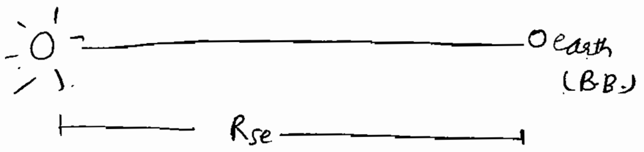
$$= \sqrt{\frac{2Z_0\,\text{Power}}{\text{Area}}}$$

$$H_0 = \frac{E_0}{Z_0} = \sqrt{\frac{2\,\text{Power}}{Z_0\,\text{Area}}}$$

E&M (11) 25/01/2012

- Pressure = $U = \left(\frac{S}{c}\right)$: Pressure $\Leftrightarrow$ Energy per unit volume.
- $= \frac{U}{3} = \left(\frac{S}{3c}\right)$

Solar Const



$R_{se} \approx 10^{11}\, m$

Solar Const. (intensity) = Energy received by a B.B. kept at a distance of $R_{sun-earth}$, per second, with no intervening atmosphere, per unit area.

$$\boxed{S}\, 4\pi R_{se}^2 = E = (\sigma T^4)(4\pi R_{sun}^2)$$

T: Temperature of ~~surface~~ surface of sun

E ← Energy per Unit Time

S ← Solar Const. (according to Thermodynamics)
or
Poynting Vector (according to EM theory)

$$\Rightarrow \boxed{S = \sigma T^4 \left(\frac{R_{sun}}{R_{sun-earth}}\right)^2}$$

$\boxed{1\ cal = 4.18\ J}$

$= 1.36 \times 10^3\ \text{Watt}/m^2$

$\approx 2\ cal/cm^2\ \text{minute}$ ✓ easier to remember

→ Till now we have done analysis of dielectric medium and vacuum where non-attenuated propagation of Electromagnetic wave takes place, as k is real. Now let's have a look at Conducting media.

In conducting media,

$$\boxed{\nabla^2\psi - \mu\sigma\left(\frac{\partial\psi}{\partial t}\right) - \mu\epsilon\left(\frac{\partial^2\psi}{\partial t^2}\right) = 0}$$ Attenuated wave Equation

- If $\sigma = 0$, we know, then $\psi = \psi_0 e^{-i(\vec{k}\cdot\vec{r} - \omega t)}$. Its easy
- If $\sigma \neq 0$, then $\vec{k}^{\star}$ is complex & $\psi = \psi_0 e^{-i(\vec{k}^{\star}\cdot\vec{r} - \omega t)}$

→ We are assuming this to be the solution

Put $\boxed{\vec{k}^{\star} = \vec{\alpha} + i\vec{\beta}}$

$$\Rightarrow \psi = \psi_0 e^{-\vec{\beta}\cdot\vec{r}} e^{-i(\vec{\alpha}\cdot\vec{r} - \omega t)}$$

$$\vec{\nabla} = i\vec{k}^{*}$$

$$\frac{\partial}{\partial t} = -i\omega$$

★ Note that all characteristics of EM wave will be there in attenuated wave except that Amplitude will decrease with time.

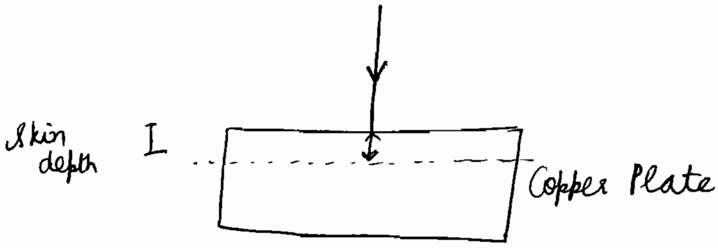
Hence using these conversions, the attenuated wave eqn becomes,

$$-k^{\star 2} + i\mu\sigma\omega + \mu\epsilon\omega^2 = 0$$

$$-(\alpha^2 - \beta^2 + 2i\alpha\beta) + i\mu\sigma\omega + \mu\epsilon\omega^2 = 0$$

$$\Rightarrow -(\alpha^2 - \beta^2 - \mu\epsilon\omega^2) + i(\mu\sigma\omega - 2\alpha\beta) = 0$$

$$\Rightarrow \alpha\beta = \left(\frac{\mu\sigma\omega}{2}\right) \quad \text{— (1)} \qquad \alpha^2 - \beta^2 = \mu\epsilon\omega^2 \quad \text{— (2)}$$



✓ Skin Depth: distance after which field falls to $\left(\frac{1}{e}\right)$ of its value in free space

$$\nabla^2\vec{E} = \mu\varepsilon\left(\frac{\partial^2\vec{E}}{\partial t^2}\right) + \mu\sigma\left(\frac{\partial\vec{E}}{\partial t}\right)$$

(for conductors)

These equations still permit plane wave solutions but this time wave number is complex

$$E_0 e^{-\beta d} = \frac{1}{e}E_0$$

$$\Rightarrow \boxed{d = \frac{1}{\beta} = \text{Skin Depth}}$$

Solving ① and ② algebraically, → using $(\alpha^2+\beta^2)^2 = (\alpha^2-\beta^2)^2 + 4\alpha^2\beta^2$, calculation is drastically reduced

$$\alpha^2 - \left(\frac{\mu\sigma\omega}{2\alpha}\right)^2 = \mu\epsilon\omega^2$$

$$4\alpha^4 - \mu^2\sigma^2\omega^2 = 4\alpha^2\mu\epsilon\omega^2$$

$$\Rightarrow \alpha^4 - \alpha^2(\mu\epsilon\omega^2) - \frac{\mu^2\sigma^2\omega^2}{4} = 0$$

$$\Rightarrow \alpha^2 = \frac{+(\mu\epsilon\omega^2) \pm \sqrt{(\mu\epsilon\omega^2)^2 + \mu^2\sigma^2\omega^2}}{2}$$

$$= \frac{\mu\epsilon\omega^2 \pm \mu\omega\sqrt{(\epsilon\omega)^2 + \sigma^2}}{2}$$

$$= \frac{\mu\epsilon\omega^2 \pm \mu\epsilon\omega^2\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}}{2}$$

$$\Rightarrow \boxed{\alpha = \left[\frac{\mu\epsilon\omega^2}{2}\right]^{\frac{1}{2}} \left[1 \pm \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}\right]^{1/2}}$$

1st expression for β

$$\beta = \frac{\mu\sigma\omega}{2}\sqrt{\frac{2}{\mu\epsilon\omega^2}}\left[1 \pm \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}\right]^{-1/2}$$

$$\boxed{\beta = \frac{\mu\sigma}{\sqrt{2\mu\epsilon}}\left[1 \pm \sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}}\right]^{-1/2}}$$

Only ⊕ve sign

✓ 2nd expression

$$\boxed{\beta = \left[\frac{\mu\epsilon\omega^2}{2}\right]^{1/2}\left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1\right]^{1/2}}$$

For good conductor, $\frac{\sigma}{\epsilon\omega} >> 1$

$$\Rightarrow \left[1 \pm \sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}}\right]^{1/2} \approx \left[1 \pm \frac{\sigma}{\epsilon\omega}\right]^{1/2} \approx \left(\frac{\sigma}{\epsilon\omega}\right)^{1/2}$$

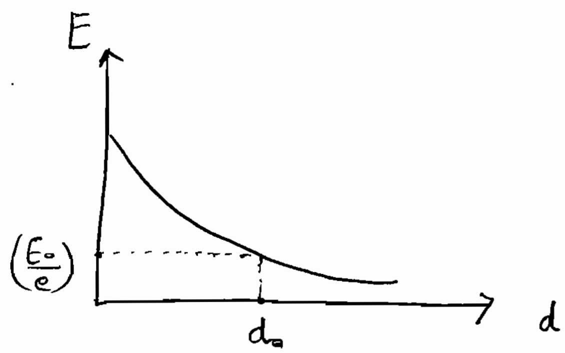
$$\Rightarrow \quad \alpha = \sqrt{\frac{\mu \epsilon \omega^2}{2}} \sqrt{\frac{\sigma}{\epsilon \omega}} = \sqrt{\frac{\mu \sigma \omega}{2}}$$

$$\beta = \frac{\mu \sigma}{\sqrt{2\mu\epsilon}} \sqrt{\frac{\epsilon\omega}{\sigma}} = \sqrt{\frac{\mu \sigma \omega}{2}}$$

$\Rightarrow$ <u>For good conductors</u>, $\boxed{\alpha = \beta = \sqrt{\frac{\mu \sigma \omega}{2}}}$

As $\sigma \to 0$, i.e. for dielectric material

$$\left.\begin{aligned} \alpha &\to \sqrt{\mu \epsilon}\, \omega = k \\ \beta &\to 0 \end{aligned}\right\} \text{Perfect}$$



For a good conductor :- d_0 : skin depth $= \frac{1}{\beta} = \sqrt{\frac{2}{\mu \sigma \omega}}$

<u>Scalar and Vector Potentials [Electromagnetic]</u>

Note that these Potentials are different from Electrostatic Potentials.

In electrostatics we studied :- $V = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$: Coulumb Potential

$dV = \int E \cdot dl$, $\vec{E} = -\vec{\nabla} V$, $\vec{\nabla} \times \vec{E} = 0$

$dV = 0 = \oint E \cdot dl$

• In electromagnetics, fields are changing w.r.t. time i.e. accelarated charge motion, hence completely different analysis, thereby giving different Potentials.

$$\boxed{\text{In Magnetostatics, we studies} \quad \vec{B} = -\vec{\nabla}\phi_m \qquad \phi_m = \frac{\mu_0 I}{4\pi}\Omega}$$

From Maxwell equations,

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = -\vec{\nabla}\times\left(\frac{\partial \vec{A}}{\partial t}\right)$$

From Biot Savert's law, $\boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$

where we define $\vec{A}$ = EM vector Potential

if valid

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int \frac{I\, d\vec{l}}{r}}$$

$$\Rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

Curl of Grad is Zero

$\Rightarrow \left[\vec{E} + \left(\frac{\partial \vec{A}}{\partial t}\right)\right]$ can be written as $-\vec{\nabla}\phi$

$$\boxed{\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi}$$

Electromagnetic ~~Scalar~~ Scalar Potential

Electromagnetic Vector Potential

$$\Rightarrow \boxed{\begin{aligned} \vec{E} &= -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \vec{\nabla}\times\vec{A} \end{aligned}}$$

<u>EM Fields in terms of Potentials $(\vec{A}, \phi)$</u>

✓ EM Potentials are mathematical constructs which are dependent on each other. They are used to measure $\vec{E}$ and $\vec{B}$. Note that $\vec{E}$ and $\vec{B}$ are dependent upon scale. unlike [redacted] potentials that are scale invariant. Physical measurement of ϕ and $\vec{A}$ is not, thereby, important as they are scale invariant, but their variation is important as it gives absolute value of $\vec{E}$ & $\vec{B}$.

<u>To prove $\vec{E}$ & $\vec{B}$ are scale invariant even if ϕ and A are changed</u>

Let $\underline{\phi = \phi' - \left(\frac{\partial \lambda}{\partial t}\right)}$ $\Rightarrow$ $\phi' = \phi + \left(\frac{\partial \lambda}{\partial t}\right)$

Let $\underline{\vec{A} = \vec{A}' - \vec{\nabla}\lambda}$ $\Rightarrow$ $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$

when λ is constant scalar quantity.

We can get at these forms

$$\begin{aligned} \vec{B}' &= \vec{\nabla}\times(\vec{A}') \\ &= \vec{\nabla}\times(\vec{A} + \vec{\nabla}\lambda) \\ &= \vec{\nabla}\times\vec{A} + \vec{\nabla}\times(\vec{\nabla}\lambda) \\ &= \vec{\nabla}\times\vec{A} \\ &= \vec{B} \end{aligned}$$

$$\begin{aligned} \vec{E}' &= -\vec{\nabla}\phi' - \frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla}\left(\phi + \frac{\partial \lambda}{\partial t}\right) - \frac{\partial}{\partial t}\left(\vec{A} + \nabla\lambda\right) \\ &= -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \\ &= \vec{E} \end{aligned}$$

Note that upon coupling 4 Maxwell equations, we get 2 equations only :

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{B} = 0$$

Recasting Maxwell equations in terms of $(\vec{A}, \phi)$:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \quad \text{— (1)}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{— (2)}$$

$$\vec{\nabla}\times\vec{E} = -\left(\frac{\partial \vec{B}}{\partial t}\right) \quad \text{— (3)}$$

$$\vec{\nabla}\times\vec{B} = \mu\left(\vec{J} + \epsilon\frac{\partial \vec{E}}{\partial t}\right) \quad \text{— (4)}$$

From (4),

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu\left(\vec{J} + \epsilon\frac{\partial}{\partial t}\left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\right)\right)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{A}(\vec{\nabla}\cdot\vec{\nabla}) + \mu\epsilon\left(\frac{\partial^2 \vec{A}}{\partial t^2}\right) + \mu\epsilon\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right) = \mu\vec{J}$$

$$\boxed{\nabla^2\vec{A} - \mu\epsilon\left(\frac{\partial^2 \vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu\epsilon\frac{\partial\phi}{\partial t}\right) = -\mu\vec{J}} \quad \text{— (5)}$$

$$\Rightarrow \underbrace{\left(\nabla^2 - \mu\epsilon\frac{\partial^2}{\partial t^2}\right)}_{\text{de Alembert}}\vec{A} - \vec{\nabla}L = -\mu\vec{J}$$

where $L = \vec{\nabla}\cdot\vec{A} + \mu\epsilon\left(\frac{\partial\phi}{\partial t}\right)$

$$\vec{\nabla}\cdot\left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\right) = \frac{\rho}{\epsilon_0}$$ From ①

$$-\nabla^2\phi - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \frac{\rho}{\epsilon_0}$$

Writing again in the same form as ⑤ by adding & subtracting $\mu\epsilon\frac{\partial^2\phi}{\partial t^2}$

$$\nabla^2\phi - \underline{\mu\epsilon\left(\frac{\partial^2\phi}{\partial t^2}\right) + \mu\epsilon\left(\frac{\partial^2\phi}{\partial t^2}\right)} + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\epsilon_0}$$

$$\Rightarrow \boxed{\left(\nabla^2 - \mu\epsilon\frac{\partial^2}{\partial t^2}\right)\phi + \frac{\partial}{\partial t}\left[\vec{\nabla}\cdot\vec{A} + \mu\epsilon\frac{\partial\phi}{\partial t}\right] = -\frac{\rho}{\epsilon_0}} \quad - ⑥$$

$$\Rightarrow \left(\nabla^2 - \mu\epsilon\frac{\partial^2}{\partial t^2}\right)\phi + \frac{\partial L}{\partial t} = -\left(\frac{\rho}{\epsilon_0}\right)$$

Lorentz Gauge Equation : $L = \vec{\nabla}\cdot\vec{A} + \mu\epsilon\left(\frac{\partial\phi}{\partial t}\right) = 0$

$$\Rightarrow \boxed{\begin{aligned} \Box^2\vec{A} &= -\mu\vec{J} \\ \Box^2\phi &= -\frac{\rho}{\epsilon_0} \end{aligned}} \quad — ⑦$$

<u>Lorentz Gauge Condition is Gauge Invariant</u> i.e.

$$\underline{\vec{\nabla}\cdot\vec{A} + \mu\epsilon\left(\frac{\partial\phi}{\partial t}\right) = 0 = \vec{\nabla}\cdot\vec{A}' + \mu\epsilon\left(\frac{\partial\phi'}{\partial t}\right) = 0}$$

We can always <u>Choose</u> λ s.t. $\nabla^2\lambda - \mu\epsilon\left(\frac{\partial^2\lambda}{\partial t^2}\right) = 0$

st. $\Rightarrow \underline{\underline{\Box^2\lambda = 0}}$

From (1)

$$\nabla^2\phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\epsilon_0}$$

If I choose $\vec{\nabla}\cdot\vec{A} = 0$, it become Poisson Equation of Electrostatic & ϕ becomes V

$$\boxed{\vec{\nabla}\cdot\vec{A} = 0 \;:\; \text{Coulumb Gauge Condition}}$$

Changing EM scalar Potential to Electrostatic scalar Potential (Coulumb Potential)

Writing (5) and (6) under Coulumb Gauge Condition

$$\boxed{\begin{aligned} \Box^2\vec{A} - \mu\epsilon\,\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right) &= -\mu\vec{J} \\ \nabla^2\phi &= \frac{-\rho}{\epsilon_0} \end{aligned}} \quad \text{—— (8)}$$

In relativistic domain,

$$\boxed{\begin{aligned} J_\mu &= (\vec{J},\ ic\rho) \\ A_\mu &= \left(\vec{A},\ \frac{i\phi}{c}\right) \\ \Box^2 A_\mu &= -\mu_0 J_\mu \end{aligned}}$$

[only lorentz gauge is gauge invariant]

$$\because \quad \Box^2\vec{A} = -\mu_0\vec{J}$$

$$\Box^2\frac{i\phi}{c} = -\frac{\rho\, i}{c\epsilon_0}$$

from lorentz gauge

Potentials

Q/ Why do we do the crappy Potentials ?

A/ By the help of these potentials, we seek that how the sources (ρ and $\vec{J}$) generate electric and magnetic fields, in other words, we seek the general solution of the Maxwell equations. Given $\rho(r)$ and $\vec{J}(r,t)$ what are the fields $\vec{E}(r,t)$ and $\vec{B}(r,t)$?
In the static case, Coulumb's law and the Biot savert's law provide the answers. What we are looking for, then, is the generalization of those laws to time-dependent configurations.

Q/ Definition of Potentials ?

A/ In electrostatics $\nabla \times \vec{E}$ was 0 and that allowed us to write $\vec{E}$ as $-\vec{\nabla} V$ but here curl is non zero. But here we still have $\nabla \cdot \vec{B} = 0$

$\Rightarrow$ We can still write

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}} \quad \text{——} (1)$$

$$\Rightarrow \vec{\nabla} \times \vec{E} = -\left(\frac{\partial B}{\partial t}\right) = -\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t}$$

$$\Rightarrow \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

Now, this quantity $\vec{E} + \left(\frac{\partial \vec{A}}{\partial t}\right)$, instead of $\vec{E}$ alone, satisfies conduction for being gradient of a potential

$$\Rightarrow \boxed{\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} \phi} \quad \text{——} (2)$$

Now from remaining two maxwell equations,

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \boxed{\nabla^2 \phi + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\frac{\rho}{\varepsilon_0}} \quad \text{--- (3)}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \left(\frac{\partial \vec{E}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} - \mu_0 \varepsilon_0 . \vec{\nabla}\left(\frac{\partial \phi}{\partial t}\right) - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\Rightarrow \boxed{\left(\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\nabla \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial \phi}{\partial t}\right) = -\mu_0 \vec{J}}$$

$$\text{--- (4)}$$

Now (3) and (4) represent all the information of Maxwell Equations in terms of Potentials.

Note that we have reduced the problem of find. 6 variables (3 components each of $\vec{E}$ & $\vec{B}$) to 4 variables (ϕ and $\vec{A}$)

Gauge Transformations

In order to further simplify the analysis, we note that equations (1) and (2) do not uniquely define the potentials; we are free to impose extra conditions on ϕ and $\vec{A}$, as long as, nothing happens to $\vec{E}$ and $\vec{B}$, thereby simplifying our analysis.

Suppose we have two sets of potentials $(\phi, \vec{A})$ and $(\phi', \vec{A}')$, that correspond to same electric and magnetic fields: By how much can they differ?

let $A' = A + \alpha$ $\qquad \phi' = \phi + \beta$

Since $\vec{B}$ is same

$$\nabla \times \vec{A} = \nabla \times \vec{A} + \nabla \times \alpha \quad \Rightarrow \quad \nabla \times \alpha = 0$$

$$\Rightarrow \text{let } \boxed{\alpha = \nabla \lambda}$$

Since $\vec{E}$ is same,

$$-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi' - \frac{\partial \vec{A}'}{\partial t}$$

$$\Rightarrow 0 = \vec{\nabla}\left(\beta + \frac{\partial \lambda}{\partial t}\right)$$

$$\Rightarrow \boxed{\beta = -\left(\frac{\partial \lambda}{\partial t}\right)}$$

$$\Rightarrow \vec{A}' = A + \vec{\nabla}\lambda$$

$$\phi' = \phi - \left(\frac{\partial \lambda}{\partial t}\right)$$

This implies that for any scalar function λ, we can add $\vec{\nabla}\lambda$ to $\vec{A}$, provided we simultaneously subtract $\left(\frac{\partial \lambda}{\partial t}\right)$ from ϕ. None of this will affect the physical quantities $\vec{E}$ and $\vec{B}$. Such changes in ϕ and $\vec{A}$ are called Gauge Transformations. They can be exploited to adjust the divergence of $\vec{A}$, with a view to simplify ugly equations 3 and 4

There are two famous gauges in physics :

Coulumb Gauge and Lorentz Gauge

Coulomb Gauge

As done in magnetostatics, put $\boxed{\vec{\nabla} \cdot \vec{A} = 0}$

This simplifies (3) to simple Poisson equation in electrodynamics

$$\nabla^2 \phi = -\left(\frac{\rho}{\varepsilon_0}\right)$$

$$\phi(r, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}', t)}{(r - r')} \, d\tau'$$

But we shouldn't be too happy as ϕ won't give $\vec{E}$ (we have to know $\vec{A}$ as well).

Also the (4) equation still remains quite ugly.

Lorentz Gauge

Here we pick $\boxed{\vec{\nabla} \cdot \vec{A} = -\mu_0 \varepsilon_0 \left(\frac{\partial \phi}{\partial t}\right)}$ (5)

The virtue of Lorentz Gauge is that it puts ϕ and $\vec{A}$, both on an equal footing.

$$\Box^2 \phi = \frac{-\rho}{\varepsilon_0} \quad \text{— (6)}$$

$$\Box^2 \vec{A} = -\mu_0 \vec{J} \quad \text{— (7)}$$

Lorentz Gauge is usually dealt with.

Retarded Potentials

In static case, Lorentz gauge equations will lead to

$$\nabla^2 \phi = \frac{-\rho}{\varepsilon_0} \qquad \nabla^2 \vec{A} = -\mu_0 \vec{J}$$

[ρ and $\vec{J}$ are independent of time]

These have standard solutions,

$$\phi(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r')}{r-r'} \, d\tau'$$

$$\vec{A}(r) = \frac{\mu_0}{4\pi} \int \frac{J(r')}{r-r'} \, d\tau'$$

However in non static case, as we know electromagnetic news travels at the speed of light, its not the status of sources right now, but rather their conditions at some time earlier, called the retarded time $\left[t - \left(\frac{|r-r'|}{c} \right) \right]$ when the message left. So we can safely conclude (we can verify that they satisfy (6) and (7)) that,

$$\phi(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho\left(r', \frac{ct - r}{c}\right)}{r} \, d\tau' \qquad r = |\vec{r} - \vec{r}'|$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}\left(\vec{r}', \frac{ct - r}{c}\right)}{r} \, d\tau'$$

Because integrands are evaluated at the retarded time, they are called retarded potentials.

<u>A beautiful thing</u>

Incidently, the advanced potentials also satisfy equation (6) and (7), i.e. $\rho\left(\vec{r}', t + \frac{r}{c}\right)$ & $\vec{J}\left(\vec{r}', t + \frac{r}{c}\right)$. This is because the d'Alembertian involves t^2 (as opposed to t), the theory itself is time-reversal invariant. And does not distinguish "past" from "future". Although the advanced potentials are entirely consistent with Maxwells equations, they violate the most sacred tenet in all of the physics : THE PRINCIPLE OF CAUSALITY. They suggest that the potentials "now" depend on what the charge and the current distribution "will" be at some point in future - the effect precedes the cause. Therefore, although the advanced potentials are of some theoretical interest, they have no direct, yet known, physical significance.

So therefore as we have calculated ϕ as well as $\vec{A}$, we can with the help of some mathematics, we are in a position to find out $\vec{E}$ and $\vec{B}$.

E&M (12) 27/01/2012

- In Lorentz gauge condition, $\Box^2 A = -\mu_0 \vec{J}$

$$\Box^2 \phi = -\frac{\rho}{\epsilon_0}$$

Note that ρ is source of electric field and $\vec{J}$ is the source of magnetic field.

In a source free region, $\rho = 0$, $\vec{J} = 0$

we have $\Box^2 \vec{A} = 0$

$$\Box^2 \phi = 0$$

Retarded Potentials

- looking for solutions of Maxwell equations in potential form: $\Box^2 \vec{A} = -\mu_0 \vec{J}$

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}\left(r', t-\frac{R}{c}\right)}{R}\, dV$$

(Solution of the eqn)

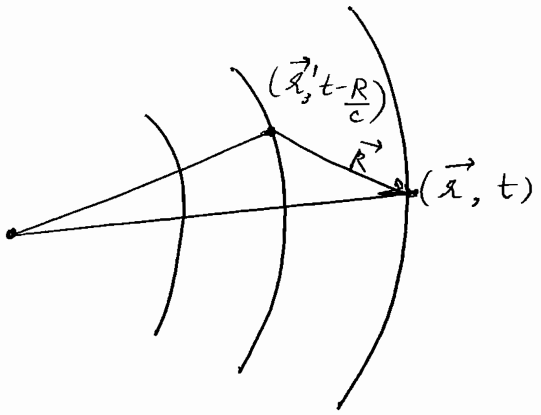
$$= \frac{\mu_0}{4\pi} \int \frac{I\left(r', t-\frac{R}{c}\right)}{R}\, d\vec{l}$$

where $R = |\vec{r} - \vec{r}\,'|$

$$\Box^2 \phi = -\frac{\rho}{\epsilon_0}$$

$$\vec{\phi}(r,t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho\left(r', t-\frac{R}{c}\right)}{R}\, dV$$

Note that $\vec{E}$, $\vec{B}$ at (r,t) are not due to present positions of charge & current. Present values of Potentials are due to past values of charge, current. Hence solutions are called retarded POTENTIALS.



Solutions are like if wavefront of EM wave is travelling. Present values of $\vec{A}$ and ϕ are due to past positions of current & charge respectively, i.e. field parameters $(\vec{E}, \vec{B})$ are not instantaneously set up.

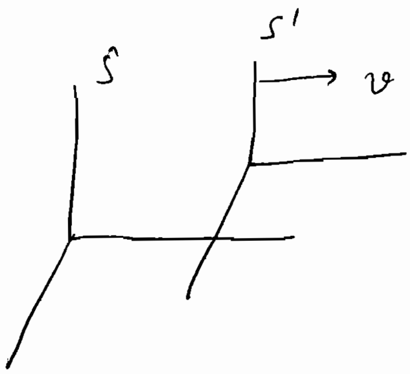
Note that if we are given $\vec{A}$, in order to calculate $\vec{J}$ we can use

$$\boxed{\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial \vec{A}}{\partial t^2} = -\mu_0 \vec{J}}$$

But we need to mention that it is the solution under Lorentz gauge condition.

Covariance of EM waves

From STR, we know



$$x_\mu = (x, y, z, ict)$$

$$\alpha_{\mu\nu} = \begin{bmatrix} \alpha & 0 & 0 & i\alpha\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\alpha\beta & 0 & 0 & \alpha \end{bmatrix}$$

where $\alpha = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ $\qquad \beta = \left(\dfrac{v}{c}\right)$

- q is lorentz invariant
- Charge density is not invariant

- Consider a stationary charge distⁿ in frame S

In this frame, since the charge is stationary, it will appear as charge distⁿ but in S' it will appear in motion and will appear as current distribution.

We have 2 new 4-vectors:

1) $\vec{J}_\mu = (\vec{J}, ic\rho)$

2) $\vec{A}_\mu = \left(\vec{A}, i\frac{\phi}{c}\right)$

$J^2 - c^2\rho^2$: invariant

$A^2 - \frac{\phi^2}{c^2}$: invariant

From equation of continuity, we know,

$$\vec{\nabla}\cdot\vec{J} + \left(\frac{\partial\rho}{\partial t}\right) = 0$$

i.e. writing in 4 coordinates

$$\frac{\partial J_1}{\partial x_1} + \frac{\partial J_2}{\partial x_2} + \frac{\partial J_3}{\partial x_3} + \frac{\partial\rho \times ic}{\partial t \times ic} = 0$$

$(x_4 = ict)$
$(J_4 = ic\rho)$

$$\Rightarrow \boxed{\frac{\partial J_1}{\partial x_1} + \frac{\partial J_2}{\partial x_2} + \frac{\partial J_3}{\partial x_3} + \frac{\partial J_4}{\partial x_4} = 0}$$

$$\boxed{\sum_{\mu=1}^{4} \frac{\partial J_\mu}{\partial x_\mu} = 0}$$ Continuity Equation

We know $\vec{J} = \rho\vec{v}$

This continuity equation is invariant.

This means, even from S',

$$\sum_{\mu=1}^{4} \frac{\partial J_\mu'}{\partial x_\mu'} = 0$$

Charge sets electric field in its own frame and magnetic field in the other frame.

$$d J_\mu' = \sum_\nu \alpha_{\mu\nu}\, d J_\nu$$

$$d x_\mu' = \sum_\nu \alpha_{\mu\nu}\, d x_\nu$$

$$\left[\frac{d J_\mu'}{d x_\mu'} = \frac{d J_\mu}{d x_\mu} \right. \Rightarrow$$ Invariance of continuity equation

To prove invariance of $\vec{J_\mu}$

$\text{Norm}(\vec{J_\mu}) = J^2 - c^2\rho^2$

$\Rightarrow$ we need to show $J_1^2 - c^2\rho^2 = J_1'^2 - c^2\rho'^2$

To prove invariance of Lorentz Gauge Condition

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} + \frac{\partial\left(\frac{i\phi}{c}\right)}{\partial(ict)} = 0$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{A} + \frac{\partial\left(\frac{i\phi}{c}\right)}{\partial x_4} = 0} \Longrightarrow \sum_{\mu=1}^{4} \frac{\partial A_\mu}{\partial x_\mu} = 0$$

define $\vec{A_\mu} = \left(\vec{A}, \frac{i\phi}{c}\right)$

hence Lorentz invariant

$\vec{A}, \phi$ are not independent

$$\Box^2 \vec{A} = -\mu_0 \vec{J} \quad \text{— (1)}$$

$$\Box^2 \phi = -\rho/\epsilon_0$$

$$\Rightarrow \Box^2\left(\frac{i\phi}{c}\right) = -\frac{i\rho}{c\epsilon_0}\frac{\mu_0}{\mu_0} = -\mu_0 (ic\rho)$$

$$\Box^2 A_4 = -\mu_0 J_4 \quad \text{— (2)}$$

Combining (1) and (2),

$$\boxed{\Box^2 A_\mu = -\mu_0 J_\mu}$$

Tensor Form of ~~Maxwell~~ Equations

Note that $\Box^2$: lorentz invariant $\Box^2 = \Box'^2$

$$\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

$$= \nabla^2 + \frac{\partial^2}{\partial (ict)^2} = \sum_{\mu=1}^{4} \frac{\partial^2}{\partial x_\mu^2}$$

Covariance of Maxwell Equations

Maxwell Equations in frame S are represented in tensor form. If they retain their form in S', then Maxwell equations are covariant.

Given $\Box^2 A_\mu = -\mu_0 J_\mu$

To prove $\Box'^2 A'_\mu = -\mu_0 J'_\mu$

We know $\left\{ \begin{array}{l} A_\mu' = \alpha_{\mu\nu} A_\nu \\ \Box'^2 = \Box^2 \end{array} \right\}$

$$\Box^{2\prime} A_\mu' = \Box^2 (\alpha_{11} A_1 + \alpha_{14} A_4)$$

$$= \alpha_{11} \Box^2 A_1 + \alpha_{14} \Box^2 A_4$$

$$= +\alpha_{11} (-\mu_0 J_1) + \alpha_{14} - \mu_0 J_4$$ (Using Tensor form)

$$= -\mu_0 [\alpha_{11} J_1 + \alpha_{14} J_4]$$

$$= -\mu_0 J_1'$$

Similarly prove for rest.

$$\Box^{2\prime} A_2' = -\mu_0 J_2'$$

$$\Box^{2\prime} A_3' = -\mu_0 J_3'$$

$$\Box^{2\prime} A_4' = -\mu_0 J_4'$$

Hence $\boxed{\Box^{2\prime} A_\mu' = -\mu_0 J_\mu'}$ ✓

So 5 things we have shown :-

1) equation of continuity is invariant
2) lorentz gauge condition is invariant
3) maxwell equations are invariant
4) $\Box^2$ is invariant
5) J_μ is a 4 vector

(1) Charge is relativistically invariant. If charge was affected by motions then we would not have exact cancellation of electronic and nuclear charge in atoms like Hydrogen or Helium.

$$\Rightarrow \int_A E \cdot dA = \int_{A'} E' \, dA'$$

(2) Surface Charge density & Volume Charge density

$$\rho' = \frac{\rho}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \sigma' = \frac{\sigma}{\sqrt{1-\frac{v^2}{c^2}}}$$

Relativistic Electrodynamics

We can express some equations of electromagnetism using 4-dimensional forms:

(1) Continuity Equation

$$\nabla \cdot \vec{J} + \left(\frac{\partial \rho}{\partial t}\right) = 0$$

$$\Rightarrow \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} + \frac{\partial (ic\rho)}{\partial (ict)} = 0$$

$$\Rightarrow \left(\frac{\partial J_1}{\partial x_1}\right) + \left(\frac{\partial J_2}{\partial x_2}\right) + \left(\frac{\partial J_3}{\partial x_3}\right) + \left(\frac{\partial J_4}{\partial x_4}\right) = 0$$

where $\vec{J_\mu} = (J_1, J_2, J_3, ic\rho) = (\vec{J}, ic\rho)$: 4 vector current density

The quantity $\left(\frac{\partial J_\mu}{\partial x_\mu}\right)$ is 4-dimensional divergence of $\vec{J_\mu}$, represented by $\Box$

$$\boxed{\Box \cdot \vec{J_\mu} = 0}$$

(2) Lorentz Condition

$$\nabla \cdot \vec{A} + \frac{1}{c^2}\left(\frac{\partial \phi}{\partial t}\right) = 0$$

or
$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} + \frac{\partial\left(\frac{i\phi}{c}\right)}{\partial (ict)} = 0$$

$$\Rightarrow \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{\partial A_4}{\partial x_4} = 0$$

$\therefore$ $\vec{A}$ and ϕ combine to form a four vector potential or electromagnetic four potential

$$A_\mu = \left(\vec{A}, \frac{i\phi}{c}\right)$$

&
$$\boxed{\Box \cdot \vec{A_\mu} = 0}$$

(3) Wave Equation for $\vec{A}$ and ϕ

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\boxed{\Box^2 \vec{A_\mu} = -\mu_0 \vec{J_\mu}}$$

Electromagnetic Field Tensor

$F_{\mu\nu}$ is an electromagnetic field tensor

$$F_{\mu\nu} = \left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \right) \quad \text{—— (1)}$$

$$F_{\mu\nu} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix}$$

From definition (1) $F_{\mu\mu} = 0$ i.e. all diagonal elements are 0.

From definition (1) $F_{\mu\nu} = -F_{\nu\mu}$ i.e. anti-symmetric matrix

$$F_{11} = 0 = F_{22} = F_{33} = F_{44}$$

$$F_{12} = -F_{21} = \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right)$$

We thus need to find 6 components to find out whole matrix

$$F_{13} = -F_{31} = \left(\frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3} \right)$$

$$\left[\frac{(\text{दूसरा})}{(\text{पहला})} - \frac{(\text{पहला})}{(\text{दूसरा})} \right]$$

$$F_{14} = -F_{41} = \left(\frac{\partial A_4}{\partial x_1} - \frac{\partial A_1}{\partial x_4} \right)$$

$$F_{23} = -F_{32} = \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \right)$$

$$F_{24} = -F_{42} = \left(\frac{\partial A_4}{\partial x_2} - \frac{\partial A_2}{\partial x_4} \right)$$

$$F_{34} = -F_{43} = \left(\frac{\partial A_4}{\partial x_3} - \frac{\partial A_3}{\partial x_4} \right)$$

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$B_1 = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} = F_{23} \quad \text{— (1)}$$

$$B_2 = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} = F_{31} \quad \text{— (2)}$$

$$B_3 = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = F_{12} \quad \text{— (3)}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\frac{-i\vec{E}}{c} = \vec{\nabla}\left(\frac{i\phi}{c}\right) + \frac{i}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\boxed{\frac{-i\vec{E}}{c} = \vec{\nabla} A_4 + \frac{\partial \vec{A}}{\partial x_4}}$$

$$\frac{-iE_1}{c} = \frac{\partial A_4}{\partial x_1} - \frac{\partial A_1}{\partial x_4} = F_{14} \quad \text{— (4)}$$

$$\frac{-iE_2}{c} = \frac{\partial A_4}{\partial x_2} - \frac{\partial A_2}{\partial x_4} = F_{24} \quad \text{— (5)}$$

$$\frac{-iE_3}{c} = \frac{\partial A_4}{\partial x_3} - \frac{\partial A_3}{\partial x_4} = F_{34} \quad \text{— (6)}$$

eg. $\underline{B_1'} = F_{23}'$

$$= \sum_{p,q}^{4} a_{2p}\, a_{3q} F_{pq}$$

$$= a_{22}\, a_{33}\, F_{23}$$

$$= \underline{B_1}$$

★ Now transformation is defined as

$$F'_{xy} = \sum_{p,q} a_{xp}\, a_{yq}\, F_{pq}$$

where $a_{p,q}$ are elements of Mincowsky Matrix

Replacing values in tensor,

$$F_{\mu\nu} = \begin{bmatrix} 0 & B_3 & -B_2 & \left(-\frac{iE_1}{c}\right) \\ -B_3 & 0 & B_1 & \left(-\frac{iE_2}{c}\right) \\ B_2 & -B_1 & 0 & \left(-\frac{iE_3}{c}\right) \\ \left(\frac{iE_1}{c}\right) & \left(\frac{iE_2}{c}\right) & \left(\frac{iE_3}{c}\right) & 0 \end{bmatrix}$$

- EM Field tensor is an antisymmetric tensor of rank 2.

$$\boxed{F_{\mu\nu}^{2} = 2\left[B^2 - \frac{E^2}{c^2}\right]}$$

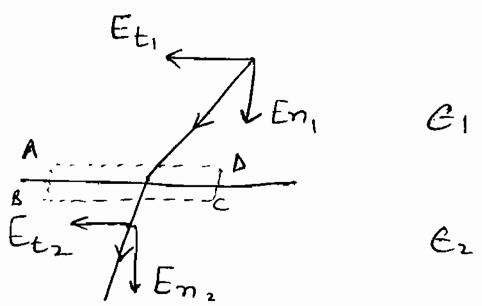
- <u>Note that using this tensor, field parameters can be written in S'.</u>

Boundary Conditions

- $\vec{D} = \epsilon\vec{E} = k\epsilon_0\vec{E}$
  $= \epsilon_0\vec{E} + \vec{P}$

- $\vec{B} = \mu\vec{H} = \mu_0\mu_r\vec{H}$
  $= \mu_0[\vec{H} + \vec{M}]$

- $\oint \vec{E}\cdot d\vec{l} = 0$
- $\oint \vec{D}\cdot d\vec{s} = q_{free}$



AB : very small

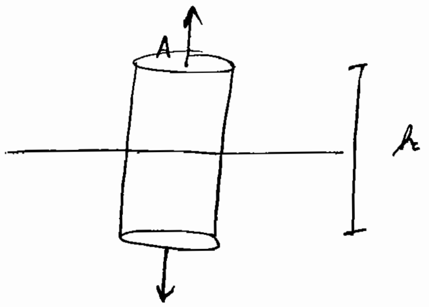
$$\oint_{ABCD} E \cdot dl = -E_{t_2}(BC) + E_{t_1}(AD) = 0$$

$$\Rightarrow \boxed{E_{t_1} = E_{t_2}}$$ Tangential Component of $\vec{E}$ is continous.

Similarly using the same,

$$\oint H \cdot dl = I$$

If there is no current on Boundary, $\boxed{H_{t_1} = H_{t_2}}$



Making $h \to 0$

$$\oint_V B \cdot ds = 0 = B_{n_1} A - B_{n_2} A = 0$$

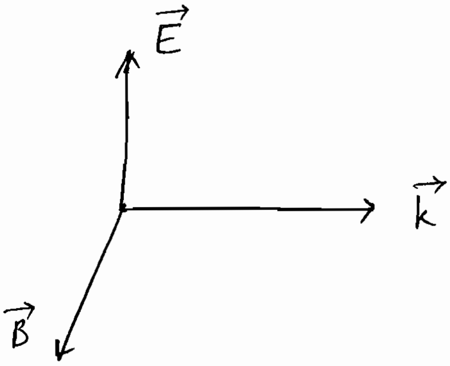
$$\Rightarrow \boxed{B_{n_1} = B_{n_2}}$$ Normal Component of $\vec{B}$ is Continuous

$$\oint D \cdot ds = q_{free}$$

If there is no charge on boundary, $\boxed{D_{n_1} = D_{n_2}}$

E&M (13)

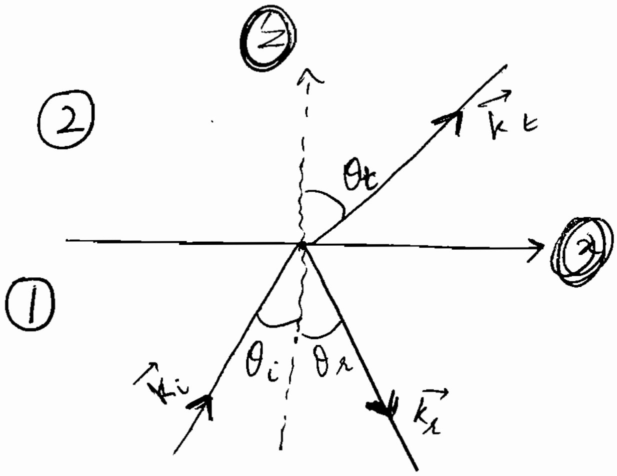
EM Waves: Reflection & Refraction



At the interface between 2 dielectric media, we saw E_T and H_T are continuous.

We know, $\left(\frac{E}{H}\right) = \left(\frac{Z_0}{n}\right) \Rightarrow H = \frac{n}{Z_0} E$

Consider interface between 2 media,



Boundary : xy plan

Plane of incidence : xz

Consider an EM wave incident at angle θ_i from normal. We want to find out reflectivity of surface i.e. fraction of energy reflected

$$\text{Reflectivity} = \left(\frac{\text{Reflected Intensity}}{\text{Incident Intensity}}\right) = \left(\frac{S_r}{S_i}\right)$$

$$\text{Transmittivity} = \left(\frac{S_t}{S_i}\right) \quad \text{where} \quad S_i = E_i^2/Z_1, \; S_r = E_r^2/Z_1, \; S_t = E_t^2/Z_2$$

R = Reflectivity = $\left(\frac{E_r}{E_i}\right)^2$

[$\vec{S}$ is proportional to n]
[From $\frac{1}{2}\varepsilon c E^2$, we have $\frac{1}{2}\varepsilon_0 n^2 \frac{c}{n} \cdot E_0^2$, also]

T = Transmittivity = $\left(\frac{Z_1}{Z_2}\right)\left(\frac{E_t}{E_i}\right)^2 = \frac{n_2}{n_1}\left(\frac{E_t}{E_i}\right)^2$

($\because \mu$ is usually μ_0, difference is mainly of ε_r)

⊙ $r = \left(\frac{E_r}{E_i}\right)$ & $t = \left(\frac{E_t}{E_i}\right)$ are given by fresnel's relations

We need to prove $R + T = 1$: Conservation of Energy
(if assumed that no energy is absorbed)

$$\boxed{R + T = \frac{E_r^2}{E_i^2} + \frac{Z_1}{Z_2}\left(\frac{E_t}{E_i}\right)^2 = 1}$$

energy per unit time reaching a particular patch of area on the surface = energy per unit time leaving the patch.

From the diagram on previous page,

$$\vec{k_i} = k_i \sin\theta_i \hat{x} + k_i \cos\theta_i \hat{z}$$
$$\vec{k_r} = k_r \sin\theta_r \hat{x} - k_r \cos\theta_r \hat{z}$$
$$\vec{k_t} = k_t \sin\theta_t \hat{x} + k_t \cos\theta_t \hat{z}$$

$$\vec{E_i} = \vec{E_{i_0}} e^{i(\vec{k_i}\cdot\vec{r} - \omega_i t)}$$
$$\vec{E_r} = \vec{E_{r_0}} e^{i(\vec{k_r}\cdot\vec{r} - \omega_r t)}$$
$$\vec{E_t} = \vec{E_{t_0}} e^{i(\vec{k_t}\cdot\vec{r} - \omega_t t)}$$

We know the boundary conditions,

$$E_{t_1} = E_{t_2}$$
$$\Rightarrow (\vec{E_I})_t + (\vec{E_R})_t = (\vec{E_T})_t \quad \text{at } z = 0 \quad \forall (x, y, t)$$

$$\Rightarrow (\vec{E_{I_0}})_t e^{i(\vec{k_i}\cdot\vec{r} - \omega_i t)} + (\vec{E_{R_0}})_t e^{i(\vec{k_R}\cdot\vec{r} - \omega_r t)}$$
$$= (\vec{E_{T_0}})_t e^{i(\vec{k_T}\cdot\vec{r} - \omega_t t)}$$

We can put $\vec{r} = 0$

If the equality is valid for all times, this can only happen when $\boxed{\omega_i = \omega_r = \omega_t}$ —— (1)

<u>Hence EM waves frequency does not change upon reflection or refraction. velocity & wavelength may change but frequency is const.</u>

Now we can put $t = 0$

If the equality is valid for all $\vec{r}$, this can only happen when (along the interface)

$$\boxed{\vec{k}_i \cdot \vec{r} = \vec{k}_r \cdot \vec{r} = \vec{k}_t \cdot \vec{r}} \quad \text{—— (2)}$$

This also implies that all the 3 waves lie in the same plane

i.e. all phase parts are same

$$\Rightarrow \boxed{(\vec{E}_{i_0})_t + (\vec{E}_{r_0})_t = (\vec{E}_{T_0})_t}$$

We have $\vec{r} = (x\hat{x} + y\hat{y} + z\hat{z})$

Laws of Reflection

$$\vec{k_i} \cdot \vec{r} = \vec{k_r} \cdot \vec{r} \quad \text{and} \quad \omega_i = \omega_r$$

$$\Rightarrow k_i x \sin\theta_i + k_i z \cos\theta_i = k_r x \sin\theta_r - k_r z \cos\theta_r$$

$\boxed{@\ z=0 \ \text{(boundary)}}$

$$k_i \sin\theta_i = k_r \sin\theta_r$$

$$\Rightarrow \boxed{\sin\theta_i = \sin\theta_r}$$

$v = \frac{\omega}{k} \Rightarrow k_i = \omega_i\sqrt{\mu_i\epsilon_i} = \omega_r\sqrt{\mu_r\epsilon_r} = k_r$

Laws of refraction

$$\vec{k_i}\cdot\vec{r} = \vec{k_t}\cdot\vec{r}$$

$$\Rightarrow \quad k_i\, x \sin\theta_i = k_t\, x \sin\theta_t \qquad (\text{at } z=0)$$

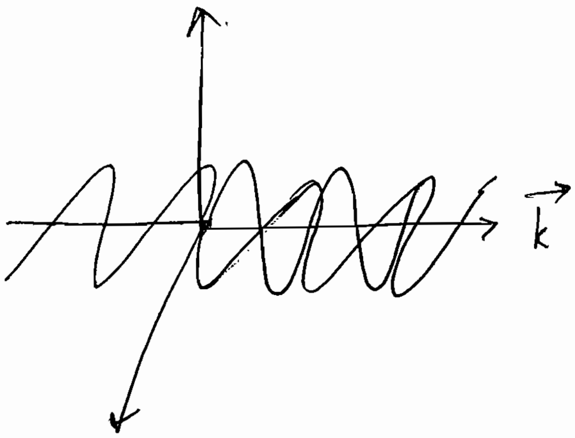
$$\Rightarrow \quad k_i \sin\theta_i = k_t \sin\theta_t$$

$$k_i = \omega_i \sqrt{\mu_i \epsilon_i}$$

$$k_t = \omega_t \sqrt{\mu_t \epsilon_t}$$

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{k_t}{k_i} = \sqrt{\frac{\epsilon_t}{\epsilon_i}} = \sqrt{\frac{\cancel{K\epsilon_t}}{\cancel{K\epsilon_i}}} = \frac{n_t}{n_i}$$

$$\Rightarrow \quad \boxed{n_i \sin\theta_i = n_t \sin\theta_t} \qquad \text{Snell's law}$$



For a given $\vec{k}$,
Now $\vec{E}$ can be polarized in 2 planes,
(i) Parallel to plane of incidence
(ii) Perpendicular to plane of incidence

Case 1 E is Parallel to plane of incidence

This implies $\vec{H}$ will lie tangential on boundary.

$$(\vec{H}_{i_0})_t + (\vec{H}_{r_0})_t = (\vec{H}_{T_0})_t$$

$$\Rightarrow \quad \boxed{H_{i_0} + H_{r_0} = H_{T_0}} \qquad [\text{all } \vec{H} \text{ are tangential only}]$$

$$\Rightarrow \frac{n_1 E_i}{Z_0} + \frac{n_1 E_r}{Z_0} = \frac{n_2 E_T}{Z_0}$$

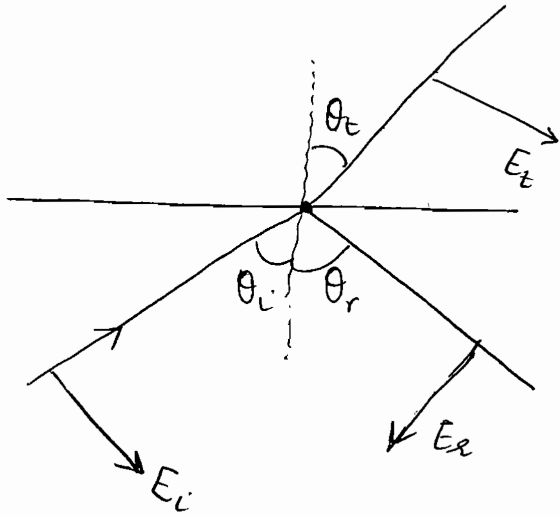
* हर case में tangential components को हम equate करते हैं !!

$$\Rightarrow \boxed{E_i + E_r = E_T\left(\frac{n_2}{n_1}\right)} \quad — (2)$$

* Use $H = \left(\frac{nE}{Z_0}\right)$

Also,

$$(\vec{E}_{i_0})_t + (\vec{E}_{r_0})_t = (\vec{E}_{T_0})_t$$



$$\Rightarrow E_i \cos\theta_i - E_r \cos\theta_r = E_t \cos\theta_t$$

$$\Rightarrow \boxed{E_i - E_r = E_t\left(\frac{\cos\theta_t}{\cos\theta_i}\right)} \quad — (3)$$

Adding (2) and (3) & Subtracting (2) and (3)

For normal incidence, $\theta_i = \theta_T = 0$

$$E_i + E_r = E_T\left(\frac{n_2}{n_1}\right)$$

$$E_i - E_r = E_T$$

Note that we can take any direction of $\vec{E}_r$

or

depending on n_1 and n_2 either E will be reversed or H.

$$\Rightarrow E_i = \left(\frac{n_1 + n_2}{2n_1}\right) E_T \quad \Rightarrow \boxed{\frac{E_T}{E_i} = \left(\frac{2n_1}{n_1 + n_2}\right)}$$

$$\Rightarrow E_r = \left(\frac{n_2 - n_1}{2n_1}\right) E_T \quad \Rightarrow \boxed{\frac{E_r}{E_i} = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)}$$

$$R + T = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2 + \frac{n_2}{n_1}\left(\frac{2n_1}{n_1 + n_2}\right)^2 = 1$$

For oblique incidence,

$$2E_i = \left(\frac{n_2}{n_1} + \frac{\cos\theta_t}{\cos\theta_i}\right) E_t \qquad \frac{n_2}{n_1} = \frac{\sin\theta_i}{\sin\theta_t}$$

$$2E_r = \left(\frac{n_2}{n_1} - \frac{\cos\theta_t}{\cos\theta_i}\right) E_t$$

$$\left(\frac{E_t}{E_i}\right) = \frac{2}{\left(\frac{\sin\theta_i}{\sin\theta_t} + \frac{\cos\theta_t}{\cos\theta_i}\right)} = \left(\frac{2\sin\theta_t \cos\theta_i}{\sin\theta_i\cos\theta_i + \sin\theta_t\cos\theta_t}\right)$$

$$= \left[\frac{4\cos\theta_i \sin\theta_t}{\sin 2\theta_i + \sin 2\theta_t}\right]$$

Fresnel equations

$$\left(\frac{E_r}{E_i}\right) = \frac{\left(\frac{\sin\theta_i}{\sin\theta_t} - \frac{\cos\theta_t}{\cos\theta_i}\right)}{\left(\frac{\sin\theta_i}{\sin\theta_t} + \frac{\cos\theta_t}{\cos\theta_i}\right)} = \left[\frac{\sin 2\theta_i - \sin 2\theta_t}{\sin 2\theta_i + \sin 2\theta_t}\right]$$

$$= \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$ ✗ (ensure that denominators are written in same form)

R + T

★ Note that factor of $\cos\theta$ will be in addition, due to oblique incidence and Energy intensity $= S \cdot \hat{z} = S\cos\theta$

$$= \left(\frac{E_r}{E_i}\right)^2 + \left(\frac{E_t}{E_i}\right)^2 \left(\frac{\sin\theta_i}{\sin\theta_t} \cdot \frac{\cos\theta_t}{\cos\theta_i}\right)$$

$$= \frac{(\sin 2\theta_i - \sin 2\theta_t)^2 + 16 \sin^2\theta_t \cos^2\theta_i \frac{\sin\theta_i}{\sin\theta_t} * \frac{\cos\theta_t}{\cos\theta_i}}{\left[\sin(2\theta_i) + \sin(2\theta_t)\right]^2}$$

$$= \underline{1}$$

✓ At grazing incidence ($\theta_i \approx 90°$), wave is totally reflected. It creates problem for those who drive @ night on wet roads.

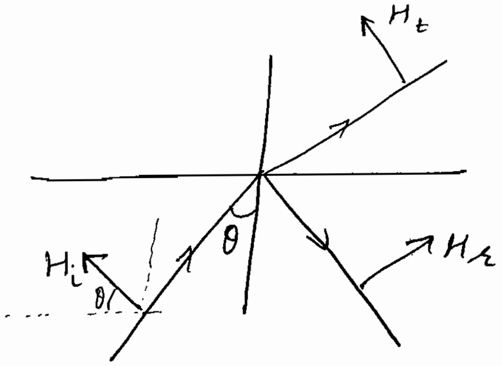
Case 2 E is perpendicular to plane of incidence.

$\Rightarrow$ E is lying on boundary

$$\Rightarrow \boxed{E_i + E_r = E_t} \quad \text{— (1)}$$

$$-H_i \cos\theta_i + H_r \cos\theta_r = -H_t \cos\theta_t$$

:- take H_i in any direction correspondingly we will have E_i and we arrive at same equation.



$$\Rightarrow \boxed{H_i - H_r = \frac{\cos\theta_t}{\cos\theta_r} H_t} \quad \text{— (2)}$$

$\Rightarrow$ using $\boxed{H = \frac{nE}{Z_0}}$

$$\boxed{E_i - E_r = \frac{n_2}{n_1}\left(\frac{\cos\theta_t}{\cos\theta_i}\right) E_t} \quad \text{— (3)}$$

Using (1) and (3),

$$\Rightarrow 2E_i = \left(1 + \frac{\sin\theta_i \cos\theta_t}{\sin\theta_t \cos\theta_i}\right) E_t \checkmark$$

$$2E_r = \left(1 - \frac{\sin\theta_i \cos\theta_t}{\sin\theta_t \cos\theta_i}\right) E_t \checkmark$$

same formula for normal incidence

$$\Rightarrow \left(\frac{E_r}{E_i}\right) = r = \left(\frac{\tan\theta_t - \tan\theta_i}{\tan\theta_t + \tan\theta_i}\right) \checkmark$$

$$\boxed{\left(\frac{E_r}{E_i}\right)_{\perp} = \frac{-\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}}$$ ✓

$$\boxed{\left(\frac{E_t}{E_i}\right)_{\perp} = \frac{2\sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)}}$$ ✓

Fresnel equations

यहाँ भी same finde:

$$\underbrace{r^2}_{R} + \underbrace{t^2 \cdot \left(\frac{\cos\theta_2}{\cos\theta_1}\right) \cdot \left(\frac{\sin\theta_1}{\sin\theta_2}\right)}_{T} = 1$$

Brewster Angle

Consider Polarization Parallel to the plane of incidence

$$\left(\frac{E_r}{E_i}\right)_{\parallel} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} = \frac{1}{\infty} = 0$$

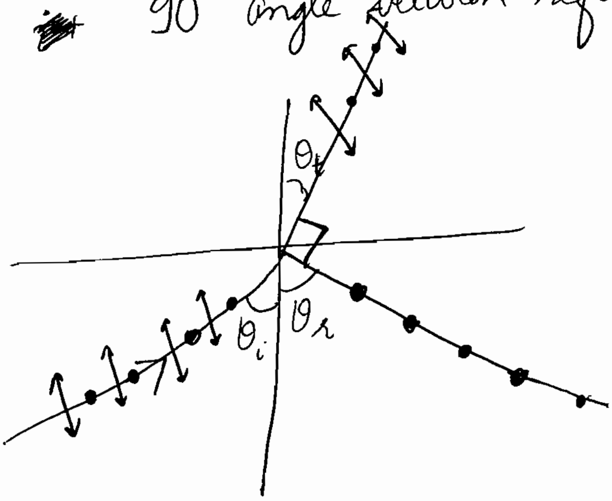
* now $\theta_1 \neq \theta_2$
$\Rightarrow$ for $\sin 2\theta_1 = \sin 2\theta_2$
$2\theta_1 = \pi - 2\theta_2$
$\Rightarrow \underline{\theta_1 + \theta_2 = \frac{\pi}{2}}$

($\vec{E}$ is $\parallel$ to plane of incident) if $\underline{\theta_i + \theta_t = 90^\circ}$: Brewster Angle

If we can make $(E_r)_{\parallel}$ zero, then we can say that light is polarized only in 1 direction. In this case, E_r is only in plane perpendicular to plane of incidence so condition for reflected wave to have vibrations in plane $\perp$ to plane of ~~motion~~ incidence:

$$\underline{\theta_i + \theta_t = (\pi/2)}$$

$\Rightarrow$ 90° angle between reflected & transmitted wave.



$\therefore$ the reflected wave is easily plane polarized as $E_{\parallel}$ component is cut off by incidence at Brewster's angle.

Total Internal Reflection

Consider Polarization to be parallel to the plane of incidence

$$\left(\frac{E_r}{E_i}\right)_{\parallel} = \frac{\left(\frac{n_2}{n_1} - \frac{\cos\theta_t}{\cos\theta_i}\right)}{\left(\frac{n_2}{n_1} + \frac{\cos\theta_t}{\cos\theta_i}\right)} = \frac{1 - \frac{n_1}{n_2}\frac{\cos\theta_t}{\cos\theta_i}}{1 + \frac{n_1}{n_2}\frac{\cos\theta_t}{\cos\theta_i}}$$

[For critical angle $n_1 \sin\theta_i = n_2$

$\Rightarrow \sin C = \left(\frac{n_2}{n_1}\right)$]

[redacted]

& $\sin\theta_t = \left(\frac{n_1}{n_2}\right)\sin\theta_i$

$$= \frac{\cos\theta_i - \frac{n_1}{n_2}\cos\theta_t}{\cos\theta_i + \frac{n_1}{n_2}\cos\theta_t} = \frac{\cos\theta_i - \frac{n_1}{n_2}\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_i}}{\cos\theta_i + \frac{n_1}{n_2}\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_i}}$$

$$= \frac{\cos\theta_i - \left(\frac{n_1}{n_2}\right)^2\sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i}}{\cos\theta_i + \left(\frac{n_1}{n_2}\right)^2\sqrt{\left(\frac{n_2}{n_1}\right)^2 \cancel{+} \sin^2\theta_i}}$$

replace θ_t by θ_i

$$= \frac{\cos\theta_i - \left(\frac{n_1}{n_2}\right)^2\sqrt{\sin^2 c - \sin^2 i}}{\cos\theta_i + \left(\frac{n_1}{n_2}\right)^2\sqrt{\sin^2 c - \sin^2 i}}$$

[For $i > c$ Put $\left(\frac{n_1}{n_2}\right)^2\sqrt{\sin^2 i - \sin^2 c} = X$]

$$= \left(\frac{\cos\theta_i - jX}{\cos\theta_i + jX}\right)$$

$\left|\frac{E_r}{E_i}\right| = 1 \Rightarrow$ Incident field is totally reflected.

DISPERSION E&M (14)

~~def^n~~ dispersion is the broadening of spectrum due to variation of n wrt. frequency of EM waves

There are 2 types of dispersion

(1) Normal Dispersion :- $\left(\frac{dv_p}{d\lambda}\right) > 0$ i.e. $\left(\frac{dn}{d\lambda}\right) < 0$

i.e. $\boxed{\left(\frac{dn}{d\omega}\right) > 0}$

(2) Anomalous dispersion :- $\left(\frac{dv_p}{d\lambda}\right) < 0$ i.e. $\left(\frac{dn}{d\lambda}\right) > 0$

i.e. $\boxed{\left(\frac{dn}{d\omega}\right) < 0}$

Now in order to calculate $(dn/d\omega)$, we know need n as $f(\omega)$

- $\vec{D} = \epsilon \vec{E} = \epsilon_r \epsilon_0 \vec{E} = \epsilon_0 \vec{E} + \vec{P}$

- $\vec{P} = (\epsilon_r - 1)\epsilon_0 \vec{E} \Rightarrow \epsilon_r = 1 + \frac{P}{\epsilon_0 E}$

We know,

$$\Rightarrow \boxed{n^2 = 1 + \frac{P}{\epsilon_0 E}}$$

Now to find n, we need $\left(\frac{P}{E}\right)$,

We know, $F = q(\vec{E} + \vec{v} \times \vec{B})$: Force exerted on charge q

This force causes displacement of dipoles which causes Polarization. Also $|\vec{B}| = \frac{|\vec{E}|}{c} \Rightarrow |\vec{E}| >> |\vec{v}||\vec{B}|$

Lorentz Force is neglected

$\Rightarrow \vec{F} = q\vec{E}$: Periodic Oscillating Force hence forced oscillations.

Particles absorb energy from EM wave and hence accelarate, due to which they radiate out energy. This radiating out energy is called Scattering.

$\vec{r}$: displacement of charged particle due to EM wave

dipole Moment $\vec{p} = e\vec{r}$

$$\vec{P} = \sum n_k \vec{p}_k$$

In a dielectric medium, restoring force acts upon the particle.

$$\Rightarrow \boxed{m\ddot{r} = -kr - b\dot{r} + qE_0 e^{i(\vec{k}.\vec{r} - \omega t)}}$$

$$\Rightarrow \ddot{r} + \frac{k}{m}r + \frac{b}{m}\dot{r} = \left[\frac{qE_0}{m} e^{i\vec{k}.\vec{r}}\right] e^{-i\omega t}$$

Using the usual symbols,

$$\ddot{r} + 2c\dot{r} + \omega_0^2 r = f_0 e^{-i\omega t}$$

if $D = \frac{d}{dt}$

$$\Rightarrow (D^2 + 2cD + \omega_0^2) r = f_0 e^{-i\omega t}$$

ω : EM wave freq.
ω_0 : molecular parameter

replacing D by $-i\omega$

$$r = \frac{f_0 e^{-i\omega t}}{(\omega_0^2 - \omega^2) - (2c\omega) i}$$

$$\boxed{r = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}} e^{-i(\omega t - \phi)}}$$

$$\boxed{\phi = \tan^{-1}\left(\frac{2c\omega}{\omega_0^2 - \omega^2}\right)}$$

eggjactly same (replace p by ω)

$\vec{p} = q\vec{x}$

$$\vec{p} = \frac{q^2 \vec{E_0}}{m} \frac{e^{i(\vec{k}\cdot\vec{x} - i\omega t)}}{(\omega_0^2 - \omega^2) \cancel{+} 2c\omega i}$$

* But natural frequency ω_0 is not the same for all vibrating molecules.

Assume n dipoles per unit volume vibrating in groups and n_k dipoles having natural frequency ω_k

$$\sum_k n_k = n$$

$$\vec{p}_k = \frac{q^2 \vec{E}}{m[(\omega_k^2 - \omega^2) - 2c\omega i]}$$

$$\vec{P} = \sum_k n_k \vec{P}_k$$

$$= \sum_k \frac{n_k q^2 \vec{E}}{m[(\omega_k^2 - \omega^2) - 2c\omega i]}$$

$$\frac{P}{E} = \frac{q^2}{m} \sum \frac{n_k}{(\omega_k^2 - \omega^2) - 2c\omega i}$$

Lorentz Dispersion Eqn

$$n^2 = 1 + \frac{P}{\epsilon_0 E} \Rightarrow \boxed{n^2 = 1 + \frac{e^2}{\epsilon_0 m} \sum \frac{n_k}{(\omega_k^2 - \omega^2) - 2c\omega i}}$$

ω_k: natural freq.
ω: EM freq

Hence $n = n(\omega)$

Case 1 Normal dispersion : damping is neglected

$$n^2 = 1 + \frac{e^2}{\epsilon_0 m} \sum \frac{n_k}{(\omega_k^2 - \omega^2)}$$

$\Rightarrow$ n becomes real

$\boxed{\text{Normal dispersion}}$ is there is regions far from natural frequencies, where damping is very small

$$n^2 = 1 + \frac{e^2}{\varepsilon_0 m} \sum \frac{n_k}{(\omega_k^2 - \omega^2)}$$

Now $\lambda = \frac{2\pi c}{\omega}$, $\lambda_k = \frac{2\pi c}{\omega_k}$

$$\Rightarrow n^2 = 1 + \frac{e^2}{m\varepsilon_0} \sum \frac{n_k \ \lambda^2 \lambda_k^2}{(\lambda^2 - \lambda_k^2)(2\pi c)^2}$$

$$= 1 + \sum_k \frac{e^2 \lambda_k^2 n_k}{m\varepsilon_0 (2\pi c)^2} \cdot \frac{\lambda^2}{\lambda^2 - \lambda_k^2}$$

$$\boxed{n^2 = 1 + \sum_k \frac{A_k \lambda^2}{\lambda^2 - \lambda_k^2}}$$

Sallmeier's Formula

where
$$A_k = \frac{n_k \cdot \lambda_k^2}{4\pi^2 c^2} \frac{e^2}{\varepsilon_0 m}$$

Now for $\lambda >> \lambda_k$, we have

$$n^2 = 1 + \sum_k A_k \cdot \frac{\lambda^2}{\lambda^2} \left[1 - \frac{\lambda_k^2}{\lambda^2}\right]^{-1}$$

$$= 1 + \sum_k \left[A_k \left(1 + \frac{\lambda_k^2}{\lambda^2} + \frac{\lambda_k^4}{\lambda^4} + \cdots\right)\right]$$

$$\boxed{n^2 = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \cdots}$$

Cauchy's Formula

$\boxed{\text{Anamalous Dispersion}}$

If frequency of EM wave lie near a natural freq, we obtain phenomenon of anomalous dispersion.

For simplicity we assume that there is only one natural frequency $\omega_k = \omega_0$.

$$n \approx 1 + \frac{e^2}{2m\varepsilon_0} \left[\frac{n}{\omega_0^2 - \omega^2 - i(2C\omega)}\right]$$

Now rationalize & put $\omega \approx \omega_0$

$$\Rightarrow \boxed{n = 1 + \frac{e^2}{2m\epsilon_0} \sum_k \left(\frac{n_k}{\omega_k^2 - \omega^2} \right)}$$ Sallmiere formula (binomial)

Sallmier's Formula : Cauchy's Formula

$$\omega_k = \frac{2\pi c}{\lambda_k}$$

$$n = 1 + \frac{e^2}{2m\epsilon_0} \sum_k \frac{n_k}{(2\pi c)^2} \left[\frac{1}{\lambda_k^2} - \frac{1}{\lambda^2} \right]^{-1}$$

$$= 1 + \frac{e^2}{(2m\epsilon_0)(2\pi c)^2} \sum n_k . \lambda_k^2 \left[1 - \left(\frac{\lambda_k}{\lambda} \right)^2 \right]^{-1}$$

Note that $\lambda_k < \lambda$ i.e. $\omega_k > \omega$. Under this damping is negligible

$$= 1 + \frac{e^2}{8\pi^2 c^2 m \epsilon_0} \sum n_k \lambda_k^2 \left[1 + \left(\frac{\lambda_k}{\lambda}\right)^2 + \left(\frac{\lambda_k}{\lambda}\right)^4 + \cdots \right]$$

[redacted]

$$\Rightarrow \boxed{n = 1 + A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4}}$$

$$\boxed{\frac{dn}{d\omega} > 0}$$

Cauchy's Formula

$$\boxed{n = 1 + A\left(1 + \frac{B}{\lambda^2}\right)}$$

Coefficient of refraction

Coefficient of dispersion

Case 2 Anomalous Dispersion

$$n^2 = 1 + \frac{e^2}{m\epsilon_0} \sum \frac{n_k}{(\omega_k^2 - \omega^2) - 2c\omega i}$$

$\Rightarrow$ n : complex

Peat $n = n_+ + i n_-$

$$n^2 = 1 + \frac{e^2}{m\epsilon_0} \sum \frac{n_k \left((\omega_k^2 - \omega^2) + i\, 2c\omega \right)}{(\omega_k^2 - \omega^2)^2 + 4c^2\omega^2}$$

$$n = 1 + \frac{e^2}{2m\epsilon_0}\sum \frac{n_k(\omega_k^2-\omega^2)}{(\omega_k^2-\omega^2)^2+4c^2\omega^2} + i\,\frac{e^2\,2c\omega}{2m\epsilon_0}\sum \frac{n_k}{(\omega_k^2-\omega^2)^2+4c^2\omega^2}$$

(Using Binomial to remove power of $\left(\frac{1}{2}\right)$)

For $\omega = \omega_K$, $n_+ = 1$ ✓

$n_- = \frac{A}{\omega}$ ✓ ; $A = \frac{2ne^2}{4c\omega\, m\epsilon_0}$ ✓

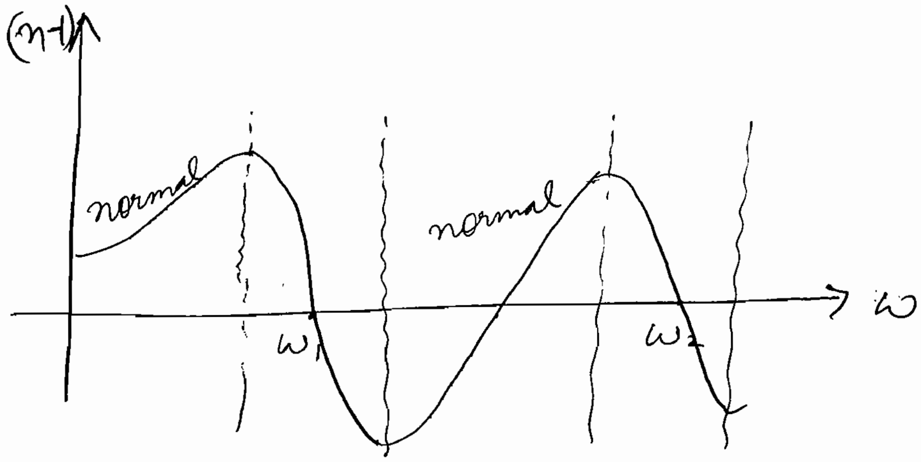
for analysis see $\omega \approx \omega_k$, $\omega \neq \omega_k$ Not equal

$\Rightarrow \left(\frac{dn}{d\omega}\right) < 0$: Anomalous dispersion

When vibration of Particles is matched by the EM frequency, maximum absorption takes place. (see the graph below)

i.e. $\frac{dn}{d\omega} < 0$: only in neighbourhood of ω_k.

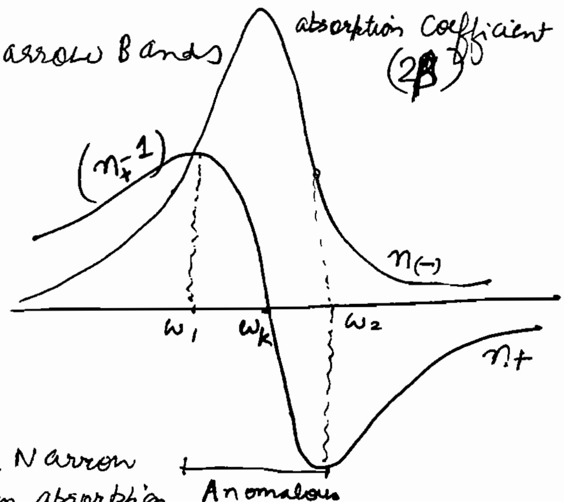
Incident frequency in neighbourhood of absorption frequency ⇒ narrow band



imaginary part leads to absorption or attenuation

Its maximum at ω_k

anomalous dispersion : Narrow Bands



ω_1 : Absorption Frequency

○ Most of the time, (index of refraction) rises gradually with increasing frequency, consistent with normal dispersion. However in immediate neighbourhood of resonance, index of refraction drops sharply, its called Anamolous Dispersion. Narrow Band of Anamolous Dispersion Coincides with maximum absorption

Scattering

Medium particles act like oscillating dipoles when EM waves are incident.

$$\vec{F} = e E_0 \sin \omega t$$

$$\vec{x} = \frac{e E_0}{m\sqrt{(\omega_k^2-\omega^2) + 4c^2 p^2}} \sin(\omega t - \theta)$$

$$\vec{p} = e\vec{x} = \frac{e^2 E_0}{m\sqrt{(\omega_k^2-\omega^2) + 4c^2 p^2}} \cdot \sin(\omega t - \theta)$$

Since its an oscillating dipole,

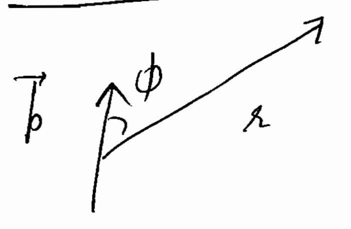
$$p = p_0 \sin(\omega t - \theta)$$

where $p_0 = \dfrac{e^2 E_0}{m\sqrt{(\omega_k^2-\omega^2)^2 + 4c^2\omega^2}}$

Dipole's Natural frequency: ω_k

Incident Frequency : ω

At $\omega_k \approx \omega$: dipole absorbs maximum energy which is radiates out : energy scattering



$$\vec{S}_{scattering} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

Scattering Energy at (r, θ) : Poynting Vector

- Scattering <
  - Rayleigh — Scattering of Bound Charged Particles. Restoring force acts. e^- do not leave ato[m]
  - Thomson — Scattering of free electrons. Nuclei treated at rest. only e^- are displaced.

$$S_{scattering} \propto \frac{P_0^2}{r^2}(1+\cos^2\phi)$$

$$S_{scattering} \propto \frac{\omega^4}{r^2}(1+\cos^2\phi)$$

$$S_{scattering} \propto \frac{1}{\lambda^4} \quad \text{(Rayleigh Scattering)}$$

---

* In case of perfect conductors, the incident wave is totally reflected with a 180° phase shift. Thats why excellent conductors make good mirrors. In practice, you paint a thin coating of silver onto the back of a pane of glass - the glass has nothing to do with the reflection; its just there is support the silver and to keep it from tarnishing. Since the skin depth of silver at optical frequencies $\approx 100\,\text{Å}$, we don't need a very thick layer.

* Since attenuated field falls off as $e^{-\beta x}$ attenuated intensity falls off as $e^{-2\beta x}$
"2β" is called Absorption Coefficient.

Note that $k = \left(\frac{\omega}{c}\right) n$

$$\Rightarrow n_{(-)} = k_{(-)}\left(\frac{c}{\omega}\right) \qquad n_{(+)} = k_{(+)}\left(\frac{c}{\omega}\right)$$

$$= \beta\left(\frac{c}{\omega}\right) \qquad = \alpha\left(\frac{c}{\omega}\right)$$

E & M (15)

02/02/2012

Induction

2 types of EM induction:

1) Self Induction : effect in same coil
2) Mutual Induction : effect on other coil

For self Induction: $\phi \propto I$

$$N\phi \propto I$$

$$N\phi = L I$$

$$\Rightarrow \boxed{L = \frac{N\phi}{I}}$$

$$-N\frac{d\phi}{dt} = -L\left(\frac{dI}{dt}\right)$$

2 definitions of induction L

$$\Rightarrow \boxed{\text{Emf } e = -L\left(\frac{dI}{dt}\right)}$$

Work Done by the circuit $= e\,i\,dt = -L\left(\frac{di}{dt}\right) i\, dt$

$$W = \int dW = \int -L\left(\frac{di}{dt}\right) i\, dt = \int -L\, i\, di = -\frac{L I^2}{2}$$

Since Work Done is ⊖ve, ⇒ Work is done upon system

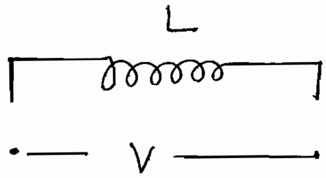
⇒ It is stored as energy.

$$\Rightarrow \boxed{\text{Energy stored} = \frac{1}{2} L I^2}$$

- In D.C. : $\frac{dI}{dt} = 0$ ⇒ No role of inductor in D.C. circuit.

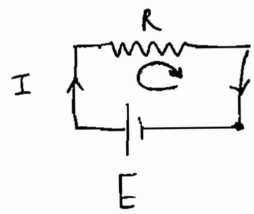
But when circuit is switched on or switched off, transcient currents flow; here inductor plays an important role.

Signs



drop in voltage $V = L\left(\frac{dI}{dt}\right)$

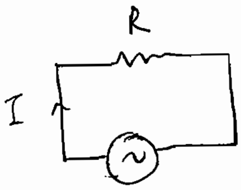
: Battery or Accumulator
It has some emf E.



$$E - IR = 0 \Rightarrow I = \left(\frac{E}{R}\right)$$

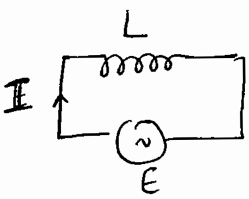
: A.C. source $\quad E = E_0 \sin \omega t = E_0\, e^{j\omega t}$

(j : used since 'i' is used for current)



$$I = \left(\frac{E}{R}\right) = \left(\frac{E_0}{R}\right) e^{j\omega t} = I_0\, e^{j\omega t}$$

(Note that phase is same)



$$L\left(\frac{dI}{dt}\right) = E$$

$$I = \frac{E_0}{L} \int e^{j\omega t}\, dt = \frac{E_0}{L} \frac{e^{j\omega t}}{j\omega}$$

$$= \frac{E_0}{j\omega L} e^{j\omega t}$$

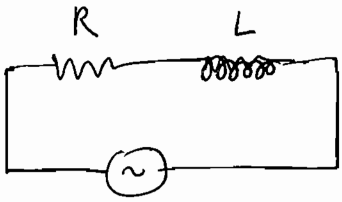
$$\boxed{Z = X_L = j\omega L}$$ Inductive Impedance or Reactance

We know $j = e^{j\pi/2}$

$$\Rightarrow I = \frac{E_0}{\omega L} \frac{e^{j\omega t}}{e^{j\pi/2}} = \frac{E_0}{\omega L} e^{j(\omega t - \pi/2)}$$

In Purely inductive circuit, current LAGS by $(\pi/2)$ wrt. EMF.

$\boxed{\text{Inductor is equivalent to resistance of value } j\omega L}$



aim is to find I, Power.

Big word is Z : impedance

It can be R : resistance

$j\omega L$: inductive reactance

$\frac{1}{j\omega C}$: capacitive reactance

In the above RL circuit

$$Z = R + j\omega L = \sqrt{R^2 + \omega^2 L^2} \angle \phi = \sqrt{R^2 + \omega^2 L^2}\, e^{j \tan^{-1}\left(\frac{\omega L}{R}\right)}$$

$\boxed{\text{We know } z = x + iy = r e^{i \tan^{-1}(y/x)} = r \angle \phi}$

$E_R = -IR$

$E_L = -L\left(\frac{dI}{dt}\right) = j\omega L I$

We know, $E_R + E_L = E$

$\Rightarrow E = IR + j\omega L I$

$\Rightarrow I = \left(E / R + j\omega L\right)$

$$IR + L\left(\frac{dI}{dt}\right) = E_0 \sin \omega t \qquad \text{— } \underline{\underline{AC}}$$

$$IR + L\left(\frac{dI}{dt}\right) = E_0 \qquad \text{— } \underline{\underline{DC}}$$

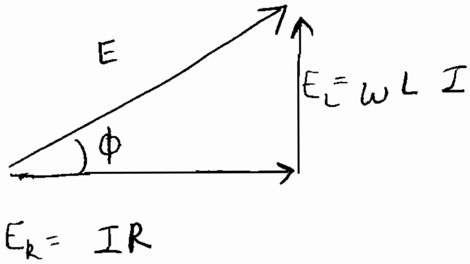
For A.C.

$$I = \frac{E_0}{\sqrt{R^2+\omega^2L^2}} \frac{e^{j\omega t}}{e^{j\phi}} = \frac{E_0\, e^{j(\omega t - \phi)}}{\sqrt{R^2+\omega^2L^2}}$$

$\Rightarrow$ Current lags by angle (or phase difference) $= \phi$

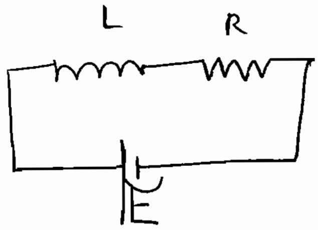
$$= \tan^{-1}\left(\frac{\omega L}{R}\right)$$

do not forget $\tan^{-1}$



$\boxed{\text{For Switching on}}$

For D.C. (Transient Response)



$$E_R + E_L = E$$

$$IR + L\left(\frac{dI}{dt}\right) = E$$

$$L\frac{dI}{dt} = E - IR$$

$$\frac{dI}{E-IR} = \frac{dt}{L}$$

$$\frac{\ln(E-IR)}{-R} = \frac{t}{L} + C_1$$

$$\Rightarrow \ln(E-IR) = -\frac{Rt}{L} - RC_1$$

At $t=0$, $I=0$ $\Rightarrow \ln(E) = -RC_1 \Rightarrow C_1 = -\frac{\ln E}{R}$

$$\Rightarrow \ln(E-IR) - \ln E = -\frac{R}{L}t$$

$$\Rightarrow e^{-\frac{Rt}{L}} = \frac{E-IR}{E} = 1 - \frac{IR}{E}$$

$$\Rightarrow \boxed{I = \frac{E}{R}\left(1-e^{-Rt/L}\right)}$$

at $t = \infty$, $I = \left(\frac{E}{R}\right)$

Power across Z = $I_z V_z$ (we know)

$\Rightarrow$ $\text{Power}_R = I^2R$: Instantaneous Power across R

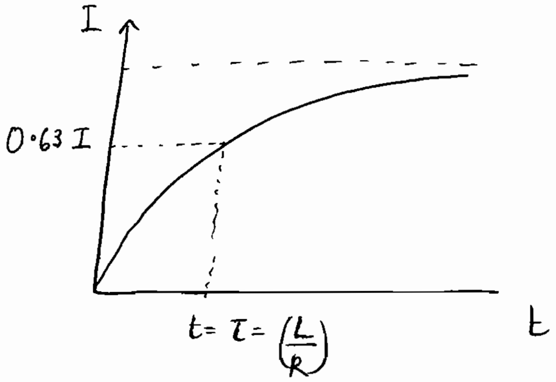
& $\text{Power}_L = IL\left(\frac{dI}{dt}\right)$: Instantaneous Power across L

$$I = I_0\left(1-e^{-Rt/L}\right)$$

at $t = \left(\frac{L}{R}\right)$ $\quad I = I_0\left(1-e^{-1}\right)$

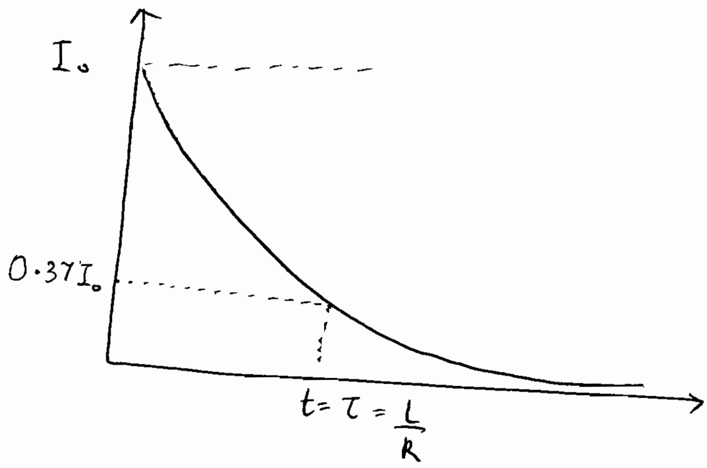
$\frac{1}{e} = 0.37 \qquad 1 - \frac{1}{e} = 0.63$

$\Rightarrow$ at $t = \tau = \frac{L}{R}$, $I = 0.63\, I_0$

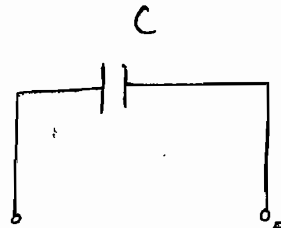


$\boxed{\text{For switching off}}$

$$IR + L\frac{dI}{dt} = 0 \qquad \boxed{I = I_0 e^{-\frac{Rt}{L}}}$$



Capacitance



$$Q = CE$$

$$I = \frac{dQ}{dt} = C\left(\frac{dE}{dt}\right)$$

$$\boxed{I = C\left(\frac{dV}{dt}\right)}$$

For A.C. $\quad I = CE_0\, j\omega e^{j\omega t} = \dfrac{E_0}{\left(\frac{1}{j\omega C}\right)} e^{j\omega t}$

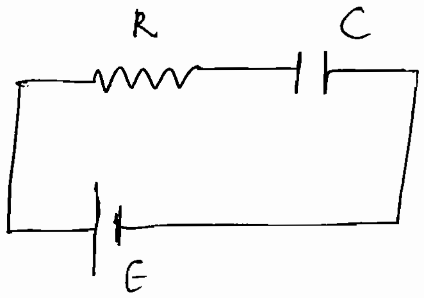
$$\boxed{Z = X_C = \frac{1}{j\omega C}}$$

For D.C., Z is infinite, hence totally block D.C except during switching on & switching off.

$$I = \frac{E_0}{\left(\frac{1}{\omega C}\right)} e^{\pi/2} e^{j\omega t} = I_0 \, e^{j(\omega t + \pi/2)}$$

⇒ For purely capacitive circuit current leads by phase $(\pi/2)$ wrt. EMF.

$\boxed{\text{Switch On}}$



$$IR + \frac{Q}{C} = E$$

$$R\frac{dQ}{dt} + \frac{Q}{C} = E$$

$$\Rightarrow \frac{dQ}{E - \frac{Q}{C}} = \frac{dt}{R}$$

$$\Rightarrow \ln\left(E - \frac{Q}{C}\right) \cdot (-C) = \frac{t}{R} + c_1$$ → do not take derivative and rather calculate Q.

$$\Rightarrow \ln\left(E - \frac{Q}{C}\right) = -\frac{t}{RC} - \frac{c_1}{C}$$

Calculate I in the end only

at $t = 0$, $Q = 0$

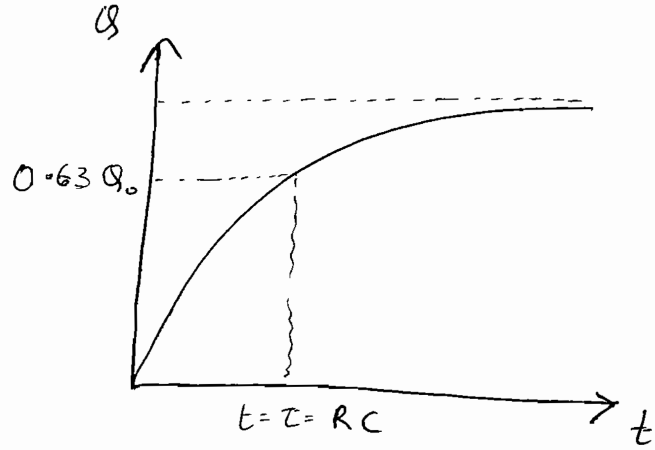
$$\Rightarrow \ln E = -\frac{c_1}{C}$$

$$\Rightarrow \ln\left(1 - \frac{Q}{EC}\right) = -\frac{t}{RC}$$

$$\Rightarrow \boxed{Q = EC\left(1 - e^{-t/RC}\right)}$$

$Q = Q_{max} = EC$ at $t = \infty$

$$I = \left(\frac{dQ}{dt}\right)$$



At $t = \tau = RC$

$$\text{Power}_R = I^2 R$$

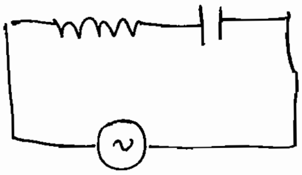
$$\text{Power}_{cap} = \frac{Q}{C}\left(\frac{dQ}{dt}\right)$$

<u>Similarly for switch off</u>

$$\boxed{Q = CV_0 \, e^{-(t/RC)}}$$

$$I_c = -\frac{V_0}{R} e^{-(t/RC)}$$

<u>R-C circuit AC</u>



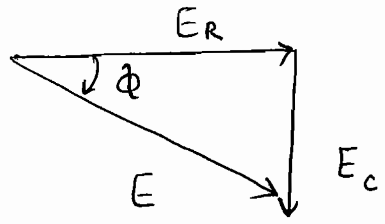
$$Z = R + \frac{1}{j\omega C} = R - \frac{j}{\omega C}$$

$$= \sqrt{R^2 + \frac{1}{\omega^2 C^2}} \; e^{-\tan^{-1}\left(\frac{1}{\omega RC}\right)}$$

$$\boxed{I = \frac{E_0}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} \, e^{j\left(\omega t + \tan^{-1}\left(\frac{1}{\omega RC}\right)\right)}}$$

$\Rightarrow$ <u>Current is leading by Phase $\tan^{-1}\left(\frac{1}{\omega RC}\right)$</u>

$$E_c = IX_c = \frac{I}{j\omega C} = \frac{I}{\omega C} e^{-j\pi/2}$$



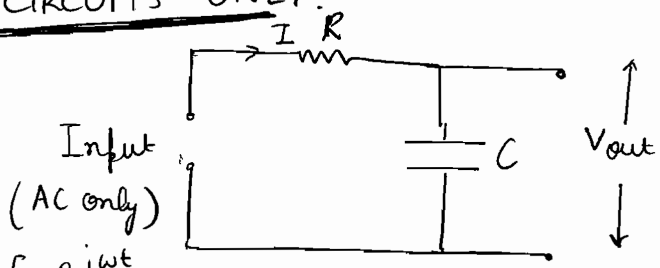
ICE
↑
Capacitor

Current 'Ice' leads Voltage
Ic'E'

$$\phi = \tan^{-1}\left(\frac{1}{\omega RC}\right)$$

Energy stored in Capacitor $= \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2$

RL and RC circuits can be used as integrator & differentiator circuits under different conditions in AC CIRCUITS ONLY.



$E = E_0 e^{j\omega t}$

INTEGRATOR

$$E_R = \frac{ER}{R} \qquad E_c = \frac{Q}{C} = \frac{1}{C}\int i\,dt = \frac{1}{RC}\int E_R\,dt$$

If $V_R = V_i \Rightarrow E_{out} = E_c = \frac{1}{RC}\int V_{In}\,dt$

Integrator Circuit output $\propto \int (\text{Input})$ For differentiator circuit output $\propto \frac{d}{dt}(\text{input})$

Our aim is to make $V_R \approx V_i$

We know $V_i = V_R + V_c$ if $V_c << V_R \Rightarrow V_i \simeq V_R$

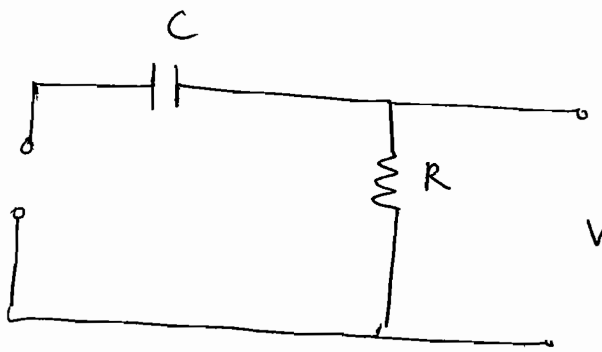
$$V_c = \frac{I}{\omega C} << IR$$

$$\Rightarrow \boxed{R >> \frac{1}{\omega C}} \quad \text{i.e.} \quad \boxed{\omega >> \frac{1}{RC}}$$

R and C are constants

hence if ω of source chosen in such a way that $\omega >> \frac{1}{RC}$

⇒ the output is integrated input.



DIFFERENTIATOR

$$V_0 = IR$$
$$= C\left(\frac{dV_c}{dt}\right) R$$
$$= RC\left(\frac{dV_c}{dt}\right)$$

$$I = \left(\frac{dq}{dt}\right), \quad E_c = \frac{q}{C}$$

$$V_o = RI = R\left(\frac{dq}{dt}\right) = RC\left(\frac{dE_c}{dt}\right) = RC\frac{dV_c}{dt}$$

$$= RC\frac{dV_i}{dt}$$

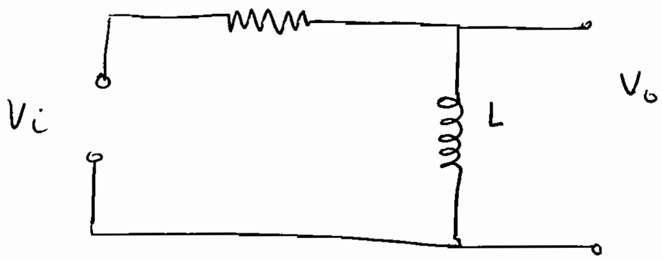
if $V_i \approx V_c$ ⇒ differentiator

$$V_i \approx V_c \text{ if } \frac{1}{\omega C} >> R$$

$$\Rightarrow \boxed{\omega << \frac{1}{RC}}$$

if output across R, and ω chosen s.t. $\omega << \frac{1}{RC}$,

circuit acts as differentiator.



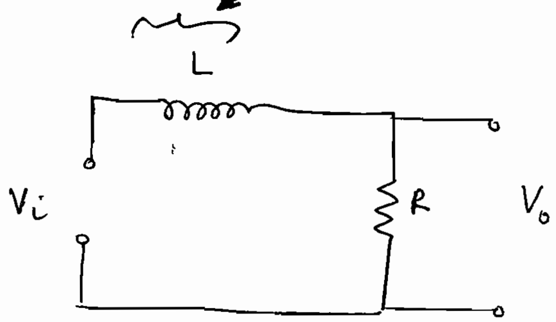
DIFFERENTIATOR

$$V_o = V_L = L\frac{dI}{dt} = L\frac{d}{dt}\frac{V_R}{R} = \frac{L}{R}\frac{d}{dt}V_R = \frac{L}{R}\frac{dV_i}{dt}$$

if $V_R \approx V_i$; if $R >> \omega L$.

$\Rightarrow$ differentiator if $\boxed{\omega << \left(\frac{R}{L}\right)}$

In all the 4 cases, we have to have a large impedance of the component placed at this position.



INTEGRATOR

$$V_o = IR = \frac{R}{L}\cdot\int V_L\, dt$$

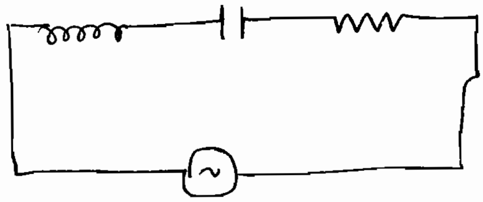
if $V_L \approx V_{in} \Rightarrow$ integrator

$\Rightarrow \omega L \gg R \Rightarrow \boxed{\omega >> \frac{R}{L}}$

→ RL circuit is choke coil, used for regulation of AC.
Its a coil of large inductance L and low resistance R.
Since Power factor $\cos\phi = \frac{R}{\sqrt{R^2+L^2\omega^2}} \approx \frac{R}{\omega L} \Rightarrow$ power absorbed by coil is very less.
Its used in AC currents for the purpose of adjusting current to any required value w/o wastage of energy.

EM (16)

LCR ckt



Acceptor Circuit

or

Series LCR Circuit

@ resonance, impedance is minimum & ckt. is called Acceptor circuit

$E = V_L + V_R + V_C$

$$\boxed{I = \frac{E}{Z} = \frac{E}{R + j\omega L + \frac{1}{j\omega C}} = \frac{E}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\, e^{j\left(\omega t - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right)}}$$

$$\phi = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$$

✓ If $\phi > 0$: inductive ckt $\quad \omega L > \frac{1}{\omega C} \Rightarrow \omega > \frac{1}{\sqrt{LC}}$

$\Rightarrow$ current lags emf

✓ If $\phi < 0$: capacitive ckt $\quad \omega L < \frac{1}{\omega C} \Rightarrow \omega < \frac{1}{\sqrt{LC}}$

$\Rightarrow$ current leads emf

✓ If $\phi = 0$: $\omega L = \frac{1}{\omega C} \Rightarrow \omega = \frac{1}{\sqrt{LC}} = \omega_0$

$\Rightarrow$ resonance : resistive ckt $\quad \phi = 0$

in phase current.

Net Reactive Component = 0

$Z = Z_{min} = R \Rightarrow$ I is maximum

$\Rightarrow$ Power delivered is maximum when frequency of source matches with frequency of circuit.

It is typically used in tuning of - radio / transistor.
By adjusting L and C, we are trying to match frequency of circuit with frequency of source.

Power Factor

Cosine of ϕ = $\cos\phi$: Power Factor

$$V = V_0 \sin \omega t$$
$$I = I_0 \sin(\omega t \pm \phi)$$

$$P = VI$$
$$= V_0 I_0 \sin(\omega t) \sin(\omega t - \phi)$$
$$= V_0 I_0 [\sin \omega t] [\sin \omega t \cos\phi - \cos \omega t \sin\phi]$$
$$= V_0 I_0 \sin^2 \omega t \cos\phi - V_0 I_0 \sin \omega t \cos \omega t \sin\phi$$

Avg. Power
$$= \frac{1}{2} V_0 I_0 \cos\phi$$

$$\boxed{P_{avg.} = V_{rms} I_{rms} \cos\phi}$$

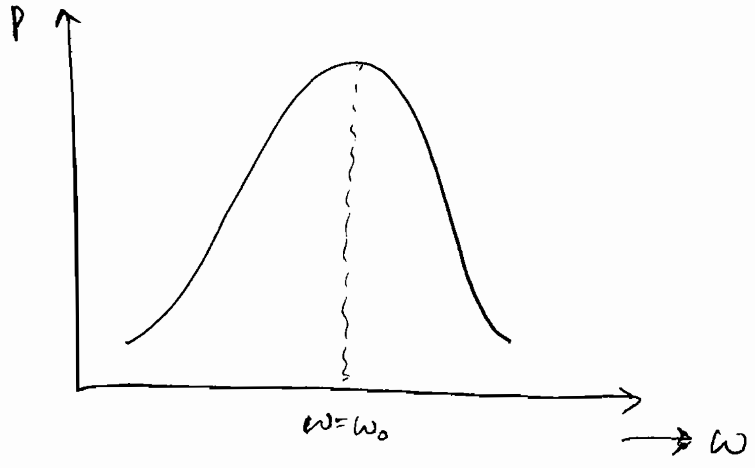
Quality Factor

In resonance $\phi = 0$, $\cos\phi = 1$

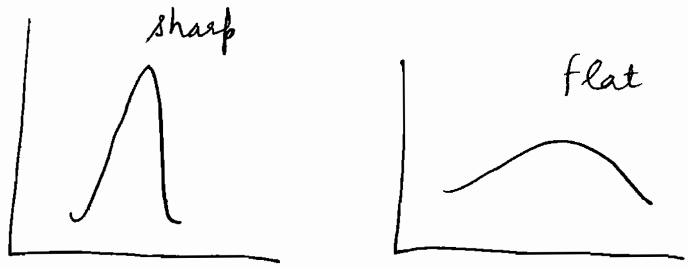
$$P_{avg} = V_{rms} I_{rms} = \frac{1}{2}(V_0)\left(\frac{V_0}{Z}\right)\left(\frac{R}{Z}\right) = \frac{V_0^2 R}{2Z^2}$$

$\Rightarrow$ $P_{avg} = 0$ for Purely [redacted] inductive or capacitive ckt.

At resonance, $P_{avg} = \frac{V_0^2}{2R}$



Quality factor is measurement of sharpness of resonance.



higher Q   lower Q

$$Q = \left(\frac{\omega_0}{\text{Bandwidth}}\right) = \left(\frac{\omega_0}{\omega_2 - \omega_1}\right)$$

where ω_1, ω_2 : half power frequencies.

$$\frac{V_0^2 R}{2\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]} = \frac{1}{2}\,\frac{V_0^2}{2R}$$

$$\left(\omega L - \frac{1}{\omega C}\right)^2 = R^2$$

$$\Rightarrow \omega L - \frac{1}{\omega C} = \pm R \quad = 0$$

$$\Rightarrow \omega^2 LC \pm R\omega C - 1 = 0 \quad \Rightarrow \omega = \frac{\pm RC \pm \sqrt{R^2C^2 + 4LC}}{2LC}$$

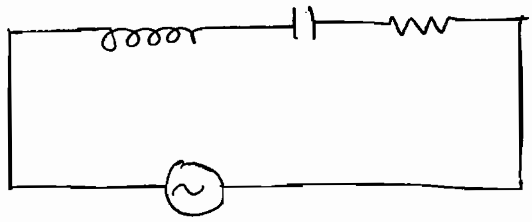
$$\boxed{Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}}$$

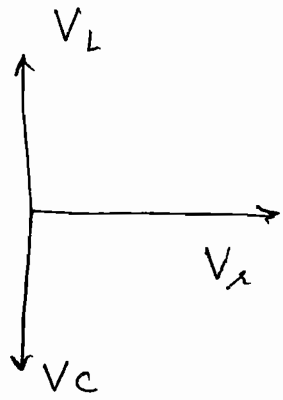
$$\Rightarrow \omega_1 = \frac{\sqrt{R^2C^2+4LC} - RC}{2LC}$$

$$\omega_2 = \frac{\sqrt{R^2C^2+4LC} + RC}{2LC}$$

$$\Rightarrow \omega_2 - \omega_1 = \frac{2RC}{2LC} = \left(\frac{R}{L}\right)$$

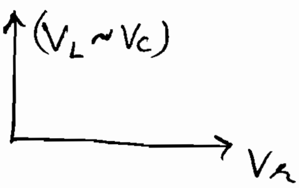
$$\Rightarrow Q = \frac{1}{\sqrt{LC}}\,\frac{2L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$



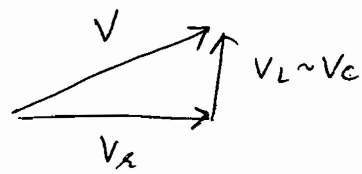


- $V_L = IX_L = jI\omega L$
- $V_C = IX_C = \frac{I}{j\omega C} = \frac{-jI}{\omega C}$
- $V_R = IR$

$\equiv$



$\equiv$



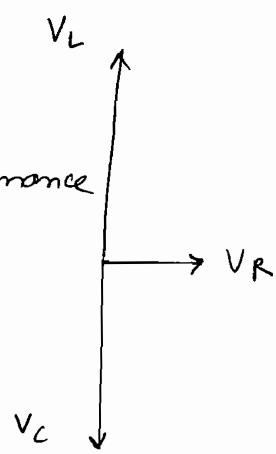
$$V = \sqrt{(V_L \sim V_C)^2 + V_R^2}$$

Quality Factor

def^1: sharpness of resonance

def^2: voltage magnifying power at resonance

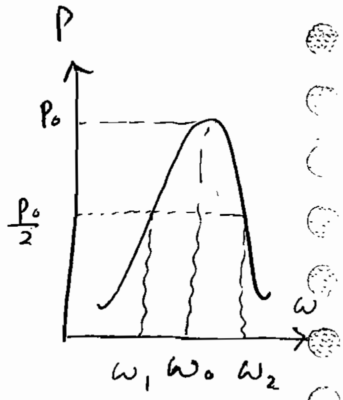
i.e. $\boxed{Q = \frac{V_L \text{ (or } V_C)}{V_{applied}} \text{ at resonance}}$



$$= \frac{I\,\omega_0 L}{IR}$$

$$= \left(\frac{\omega_0 L}{R}\right) = \frac{1}{\omega_0 CR}$$

From def^1 : $\omega_2 - \omega_1 = (R/L)$ = Bandwidth



Half Power Frequencies

$$\omega_0 - \omega_1 = \frac{R}{2L} \Rightarrow \boxed{\omega_1 = \left(\omega_0 - \frac{R}{2L}\right)}$$

$$\omega_2 - \omega_0 = \frac{R}{2L} \Rightarrow \boxed{\omega_2 = \left(\omega_0 + \frac{R}{2L}\right)}$$

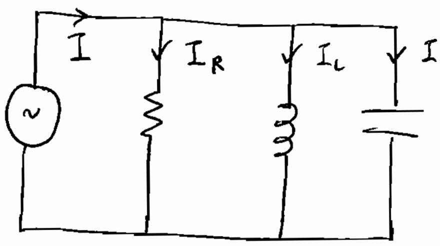
For any complex circuit, to find ω_0

Find Z_{net}

Write Z_{net} as $Z_+ + jZ_-$

Put $Z_- = 0$ and find ω_0.

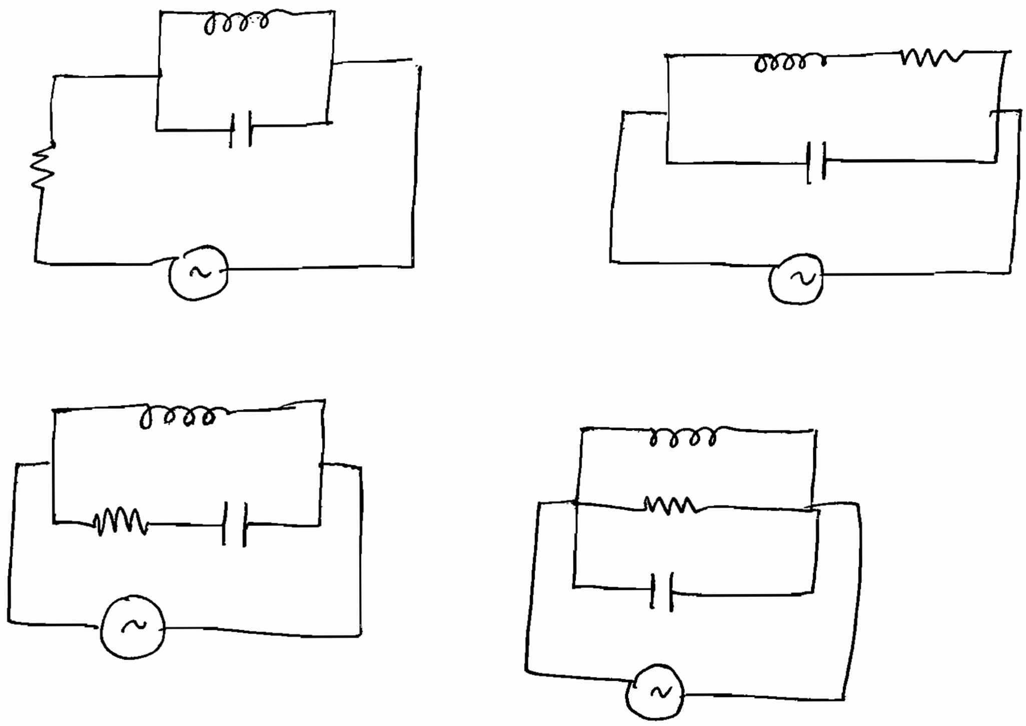
Parallel Resonant LCR.



Rejector Circuit
~~Or~~
~~Parallel LCR Circuit~~

Rejector Ckt means Current at Resonance becomes minimum while I_C and I_L becomes maximum and opposite. Hence used in filtering circuit. Use to filter out resonant frequency.

Note that parallel ckt implies only L and C are in parallel. R can be in series or parallel. But all the results are derived for the latter case.



We are interested in this

$$Y = \frac{1}{Z} \text{ is called Admittance} \ldots$$

$$I = \frac{V}{Z} = VY$$

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \Rightarrow \frac{V}{Z} = \frac{V}{Z_1} + \frac{V}{Z_2} + \frac{V}{Z_3}$$

$$\Rightarrow I = I_1 + I_2 + I_3$$

Y is written as $g + jb$

g: conductance

b: susceptance

$$\boxed{\text{At resonant condition, set } b = 0}$$

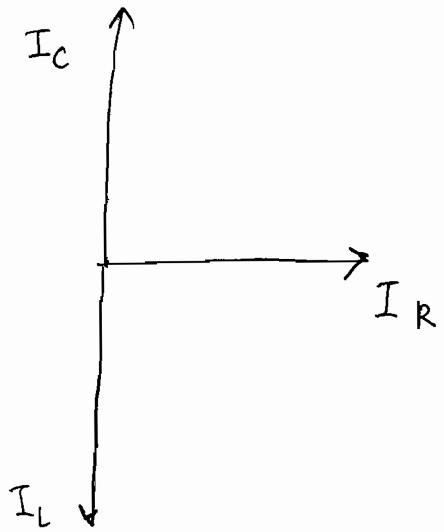
i.e. admittance is minimum at resonance

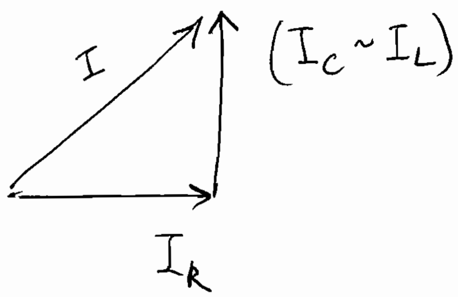
i.e. maximum impedance i.e. minimum current. Thus parallel ckt. does not allow the resonant frequency from source to pass in the circuit. Therefore, called rejector ckt. for such a freq

$$I_L = \frac{V}{j\omega L} = \frac{V_0}{\omega L} e^{j(\omega t - \pi/2)}$$

$$I_C = V j\omega C = V_0 \omega C \, e^{j(\omega t + \pi/2)}$$

$$I_R = \frac{V}{R} = \frac{V_0}{R} e^{j\omega t}$$





$$I = \sqrt{I_R^2 + (I_C \sim I_L)^2}$$

At resonance, $I_C = I_L \Rightarrow I = I_R$: minimum

Also $I_C, I_L \gg I_R$ at resonance

Hence, Quality Factor $= \left[\dfrac{I \text{ applied}}{I_L \text{ (or } I_C)}\right]_{\text{at resonance}}$: Current Magnifying Power

$$I_L = \frac{V_0}{\omega_0 L} = I_C = V_0 \omega_0 C \quad (\text{at resonance})$$

$$\Rightarrow \quad Q = \frac{V_0/R}{V_0/\omega_0 L} = \left(\frac{\omega_0 L}{R}\right) = \frac{1}{\omega_0 CR}$$

Hence same Q, same BW, same Power Factor

$$\frac{1}{Z} = \frac{1}{j\omega L} + j\omega C + \frac{1}{R} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

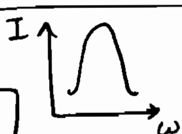
$$= g + jb$$

Set $b = 0$ for resonance

(This is because, synchronization is required for resonance. For synchronization Phase should be same. No lead or lag. Hence Put complex component which is responsible for phase = 0)

$$\Rightarrow \quad \omega_0 C = \frac{1}{\omega_0 L} \quad \Rightarrow \quad \boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

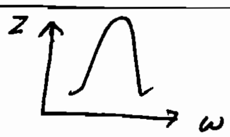
Bandwidth



(1) Series Ckt

$$I = \frac{I_0}{\sqrt{2}}$$

(2) Parallel Ckt

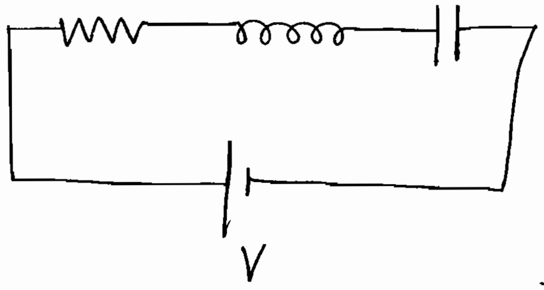


$$Z = \left(\frac{Z_0}{\sqrt{2}}\right)$$

Power in AC Ckt

True Power = Apparent Power * Power Factor

Transcient Response in LCR



→ Note that using algebraically solution is simple here. In AC it was very difficult. hence we resorted to Phasor Analysis.

$$V = V_L + V_R + V_C$$

$$V = L\left(\frac{dI}{dt}\right) + IR + \frac{q}{C}$$

→ Charge will grow across C and I will flow (increase) in L

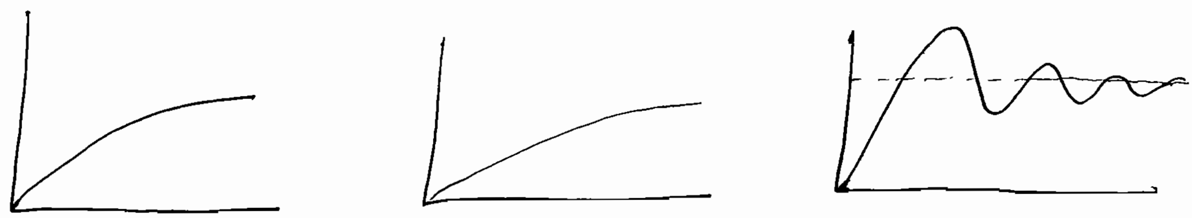
$$\Rightarrow \quad \frac{d^2I}{dt^2} + \frac{R}{L}\left(\frac{dI}{dt}\right) + \frac{I}{LC} = 0$$

$$\Rightarrow \quad \frac{d^2I}{dt^2} + 2c\left(\frac{dI}{dt}\right) + \omega_0^2 = 0$$

$$2c = \left(\frac{R}{L}\right)$$

$$\omega_0^2 = \left(\frac{1}{LC}\right)$$

Typical, control equation.
we know the solution for 3 cases.



Overdamped ($c > \omega_0$)

Critically damped ($c = \omega_0$)

$c < \omega_0$ Underdamped

$c > \omega_0$

$\underline{R > R_0}$

$c = \omega_0$

$$\frac{R}{2L} = \frac{1}{\sqrt{LC}}$$

$\underline{R = R_0}$

$c < \omega_0$

$$\Rightarrow \frac{R}{2L} < \frac{1}{\sqrt{LC}}$$

$$\Rightarrow R < 2\sqrt{\frac{L}{C}}$$

i.e. $\underline{R < R_0}$

$$\boxed{R_0 = 2\sqrt{\frac{L}{C}}}$$

$$I = I_0 e^{-\left(\frac{R}{2L}\right)t} \sin\left(\sqrt{\omega_0^2 - c^2}\, t + \phi\right)$$

$$\omega_d = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

Oscillatory circuit means that there are 2 sources of energy stored. Hence, the 2 exchange energy thereby giving rise to Oscillations.

Note that the equation written in current can be written for charge also. Hence again 3 solutions.

$$L\frac{d^2q}{dt^2} + \frac{dq}{dt}R + \frac{q}{c} = V$$

$$\Rightarrow \frac{d^2q}{dt^2} + \left(\frac{R}{L}\right)\frac{dq}{dt} + \left(\frac{q}{LC}\right) = \left(\frac{V}{L}\right)$$

Put $(q - VC) = q'$

Now we find solution for q'

Solution : (solution of L.H.S) + VC

$\Rightarrow q = q' + VC$

2011 $R_0 = 2\sqrt{\frac{10}{0.1} \times 10^3} = 200\sqrt{10}$

Or

$\omega_0 = \frac{1}{\sqrt{LC}} > C$ for oscillations

$\frac{1}{\sqrt{LC}} > \frac{R}{2L}$

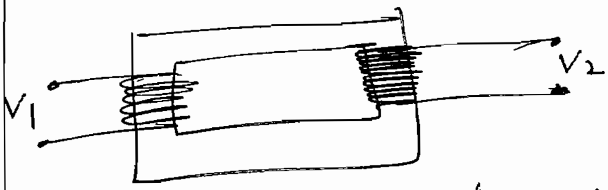
i.e. $R < 2\sqrt{\frac{L}{C}}$

$200 < 200\sqrt{10}$

Hence Oscillatory

$\omega = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$

Principle of Transformer



$\phi_1 = \phi_2$

$\Rightarrow \left(\frac{d\phi_1}{dt}\right) = \left(\frac{d\phi_2}{dt}\right)$

$\Rightarrow \frac{N_1}{N_1}\left(\frac{d\phi_1}{dt}\right) = \frac{N_2}{N_2}\left(\frac{d\phi_2}{dt}\right)$

$\Rightarrow \frac{\mathcal{E}_1}{N_1} = \frac{\mathcal{E}_2}{N_2}$

$\Rightarrow \boxed{\mathcal{E}_2 = \mathcal{E}_1\left(\frac{N_2}{N_1}\right)}$

Also

$P_1 = P_2$

$\Rightarrow \mathcal{E}_1 i_1 = \mathcal{E}_2 i_2$

$\Rightarrow i_2 = i_1\left(\frac{\mathcal{E}_1}{\mathcal{E}_2}\right)$

ज्यादा coils in secondary → step up !!

2011

Quality Factor $= 2\pi \frac{(\text{Energy Stored})}{(\text{Energy loss per cycle})}$

$q = q_0 e^{-ct} \sin(\omega_d t + \theta)$

Energy : $\frac{q^2}{2C}$

$E = \left[\frac{q_0^2}{2C}\right] e^{-2ct}$

$$\left(\frac{dE}{dt}\right) = -2cE.$$

$$Q = 2\pi \frac{E}{2cE * \tau} = \left(\frac{\omega}{2c}\right)$$

Transmission Coefficient $t = \frac{E_t}{E_i}$

while

Transmittivity $= T$

Attempt BIG QUESTIONS. Attempt those with lesser subparts.

Q10/Tut 7

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

$$= \frac{1}{R_1 + j\omega L} + \frac{1}{R_2 - \frac{j}{\omega C}}$$

$$= \frac{R_1 - j\omega L}{R_1^2 + \omega^2 L^2} + \frac{R_2 + j/\omega C}{R^2 + \frac{1}{\omega^2 C^2}}$$

$$= \left(\frac{R_1}{R_1^2 + \omega^2 L^2} + \frac{R_2}{R^2 + \frac{1}{\omega^2 C^2}}\right) - j\left(\frac{\omega L}{R_1^2 + \omega^2 L^2} - \frac{1/\omega C}{R^2 + \frac{1}{\omega^2 C^2}}\right)$$

$$= g + jb$$

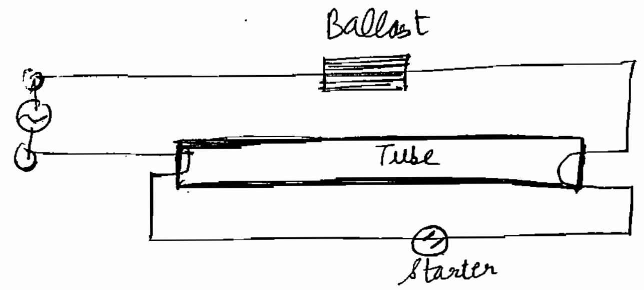
Put $b = 0$

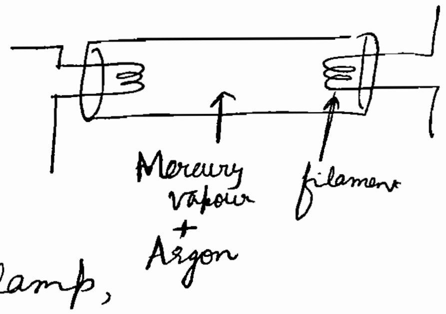
Use of choke coil in tubes

- Tubes are technically called Fluoroscent lamps. Small ones are called CFL or Compact Fluroscent lamps
- A fluoroscent lamp is a gas discharge lamp that uses Electricity to excite mercury vapour. The excited mercury vapour atoms produce short-wave UV light. UV light causes a phosphor to fluoresce, producing visible light.
- A fluoroscent lamp converts electrical power into Useful light more efficiently than an incondescent lamp. (household bulb).
- Fluoroscent lamp are negative differential resistance devices, so as more current flows through them, the electrical resistance of the fluoroscent lamp drops, allowing even more current to flow. Connected to a constant voltage power supply, a fluoroscent lamp will self destruct due to uncontrolled current flow. To prevent this, an auxillary device called BALLAST [redacted] is used. Simplest Ballast [redacted] device for AC use is an inductor placed in series, consisting of a winding on a laminated core. The inductance of this winding limits the flow of AC current. Ballast Coil is also called

Choke Coil

- Common Fluoroscent's Ballast also used an additional device called Starter. The starter is there to light the fluoroscent lamp initially.





When voltage is applied to fluorescent lamp, here's what happens :

① starter (which is simply a timed switch) allows current to flow through the filaments at the ends of the tube

② The current causes the starter's contacts to heat up & open, thus interrupting the flow of current. The tube lights

③ Since the lighted fluorescent tube has low resistance, the ballast now serves as a current limiter.

When you turn on fluorescent tube, the starter is a closed switch. The filament at the ends of the tube are heated by electricity, and they create a cloud of electrons inside the tube. The fluorescent starter is a <u>time delay switch</u> that opens afters a second or two. When it opens, the voltage across the tube allows the stream of electrons to flow across the tube and ionize the mercury vapour.

Without the starter, a steady stream of electrons is never created (as not enough e^- are produced due to low heating) between the two filaments, and the lamp flickers.

Without the ballast or choke coil, the arc is a short circuit b/w the filaments, and this short ckt. contains a lot of current. The current either vapourizes the filaments or causes the lamp to explode. Note that L is high but $\cos\phi$ is low $\Rightarrow$ low current and low wastage of power.

For a series LCR :

$$P_{delivered} = V_{rms} I_{rms} \cos\phi = \frac{V_0}{\sqrt{2}} \cdot \frac{I_0}{\sqrt{2}} \cos\phi$$

$$= \frac{V_0}{2} \cdot \frac{V_0}{Z} \cdot \frac{R}{Z} = \left(\frac{V_0^2 R}{2 Z^2}\right)$$

$$P_{consumed} = I_{rms}^2 R = \frac{I_0^2}{2} R = \left(\frac{V_0^2}{2Z^2} R\right)$$

@ Resonance & Half Power Points

$$P = \frac{P_0}{2} = \left(\frac{V_0^2 R}{4 Z^2}\right)$$

@ $I = \frac{I_0}{\sqrt{2}}$, $P = \left(\frac{I_0}{\sqrt{2}}\right)^2 \frac{R}{2} = \frac{I_0^2 R}{4} = \left(\frac{V_0^2 R}{4Z^2}\right)$

Resonance पर हो न हो, जो power deliver होगी, उसे 'R' ही खाएगा न !!

For a parallel LCR :

@ resonance : $Z = Z_{max}$

$Y = Y_{min}$

@ Bandwidth points $Z = \left(\frac{Z_0}{\sqrt{2}}\right)$

$$Y = \sqrt{2}\, Y_0$$

$\vec{\nabla} \cdot \vec{E} = 0$ $\quad$ $\vec{\nabla} \cdot \vec{B} = 0$ $\quad$ $\vec{\nabla} \times \vec{E} = -\left(\frac{\partial \vec{B}}{\partial t}\right)$ $\quad$ $\vec{\nabla} \times \vec{B} = \mu_0 \sigma \vec{E} + \mu_0 \varepsilon_0 \left(\frac{\partial \vec{E}}{\partial t}\right)$

$$\nabla \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) + \vec{\nabla}^2 \vec{E} = + \frac{\partial}{\partial t}\left(\mu_0 \sigma \vec{E} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\boxed{\nabla^2 \vec{E} = \mu_0 \sigma \left(\frac{\partial \vec{E}}{\partial t}\right) + \mu_0 \varepsilon \frac{\partial^2 E}{\partial t^2}}$$